

Generalized unitarity and the worldsheet S matrix in $\text{AdS}_n \times \mathbf{S}^n \times \mathbf{M}^{10-2n}$

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Abstract

The integrability-based solution of string theories related to $\text{AdS}_n/\text{CFT}_{n-1}$ dualities relies on the worldsheet S matrix. Using generalized unitarity we construct the terms with logarithmic dependence on external momenta at one- and two-loop order in the worldsheet S matrix for strings in a general integrable worldsheet theory. We also discuss aspects of calculations at higher orders. The S-matrix elements are expressed as sums of integrals with coefficients given in terms of tree-level worldsheet four-point scattering amplitudes. One-loop rational functions, not determined by two-dimensional unitarity cuts, are fixed by symmetry considerations. They play an important role in the determination of the two-loop logarithmic contributions. We illustrate the general analysis by computing the logarithmic terms in the one- and two-loop four-particle S-matrix elements in the massive worldsheet sectors of string theory in $\text{AdS}_5 \times \mathbf{S}^5$, $\text{AdS}_4 \times \text{CP}^3$, $\text{AdS}_3 \times \mathbf{S}^3 \times \mathbf{S}^3 \times \mathbf{S}^1$ and $\text{AdS}_3 \times \mathbf{S}^3 \times \mathbf{T}^4$. We explore the structure of the S matrices and provide explicit evidence for the absence of higher-order logarithms and for the exponentiation of the one-loop dressing phase.

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1 Introduction

The integrability of quantum string theory in $\text{AdS}_5 \times \text{S}^5$ has led to remarkable progress in our understanding of the spectrum of anomalous dimensions in the dual $\mathcal{N} = 4$ super-Yang-Mills theory. Similarly, the integrability of the spin chain with the dilatation operator of $\mathcal{N} = 4$ super-Yang-Mills theory as Hamiltonian offers important insight into the spectrum of string theory in $\text{AdS}_5 \times \text{S}^5$ and also provides tools for the construction of other important quantities, such as the correlation functions of gauge-invariant operators and four-dimensional scattering amplitudes. This remarkable success [1] raises the question of applying similar methods, such as algebraic curve techniques [2] or the Asymptotic Bethe Ansatz (ABA) [3], to other string theories that exhibit integrable structures, such as strings in $\text{AdS}_4 \times \text{CP}^3$ or in $\text{AdS}_n \times \text{S}^n \times \text{M}^{10-2n}$ supported by either an RR flux or a mixture of RR and NSNS fluxes, and shed light on the dual conformal field theories which, for $n = 3$ are not understood beyond their BPS sector. The essential ingredient in such an approach is the scattering matrix of worldsheet excitations around a suitably-chosen vacuum state or, alternatively, the scattering of spin-chain excitations.

The assumption of quantum integrability and the symmetries of the theory go a long ways in the construction of worldsheet S matrices. The former implies that the S matrices obey a form of the Yang-Baxter equation and that higher-point S-matrix elements can be constructed by multiplying together four-point S-matrix elements. The latter implies the factorization of the S matrix into factors invariant under each of the symmetry groups of the gauge-fixed theory. All in all, the S matrices are uniquely determined up to overall phases – known as the dressing phases – and a function of the string tension. There is one such phase for each part of an S matrix that is invariant under the symmetry group.

Green-Schwarz-type supercoset sigma models for $\text{AdS}_3 \times \text{S}^3$, $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3$ and $\text{AdS}_2 \times \text{S}^2$ can be constructed based on $D(2, 1; \alpha) \times D(2, 1; \alpha)/(SU(1, 1) \times SU(2) \times SU(2))$, $PSU(1, 1|2) \times PSU(1, 1|2)/(SU(1, 1) \times SU(2))$ and $PSU(1, 1|2)/(SO(1, 1) \times U(1))$, respectively. Their dimension is however smaller than $d = 10$, additional bosonic directions for a critical string theory. Unlike the NSR string, the worldsheet theory of the Green-Schwarz string is interacting even when the bosonic part of the target space is a product space with interactions induced by fermions which are representations of the ten-dimensional Lorentz group. Thus, these supercoset sigma models can be related to the Green-Schwarz string on $\text{AdS}_n \times \text{S}^n \times \text{M}^{10-2n}$ with $\text{M} = \text{S}^3 \times \text{S}^1$, T^4 and T^6 only if there exists a non-degenerate κ -symmetry gauge that decouples these additional degrees of freedom. Such a gauge, decoupling the lone S^1 , was found for $\text{M} = \text{S}^3 \times \text{S}^1$ [4]. For $\text{M} = \text{T}^4$ and T^6 , while no such gauge appears to exist, it is possible at the

classical level to consistently truncate¹ [5] all the fields orthogonal to the supercoset (which are massless). While superficially this truncation may seem inconsistent at loop level, we shall argue that the integrability of the theory implies that, through two loops, the truncated states affect only the part of the S-matrix terms whose dependence on external momenta is completely rational.

Similarly to strings in $\text{AdS}_5 \times \text{S}^5$, these theories were quantized around a BMN-type point-like string and Asymptotic Bethe Ansatz equations were proposed. An important departure from the structure of the $\text{AdS}_5 \times \text{S}^5$ string is that the free worldsheet spectrum is more complex, exhibiting excitations of different masses. Moreover, masses are conserved in the scattering process which, under certain circumstances, also forbids the exchange of momentum between particles (*i.e.* it is reflectionless). The same steps as for string theory in $\text{AdS}_5 \times \text{S}^5$ [6] led to the construction of finite-gap equations [7] and led to constraints on dressing phases.²

The S matrix of string theory in $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$ and $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$ and of the corresponding spin chain was further analyzed from an algebraic standpoint in [13, 14, 15, 16, 17, 18] where it was emphasized that, unlike the case of strings in $\text{AdS}_5 \times \text{S}^5$ [19, 20, 21], the dressing phases are not determined by crossing symmetry constraints. The one-loop (worldsheet) correction to the dressing phases in $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$ were found in [22] (see also [13]) through a comparison of the one-loop correction to the energy of certain extended string configuration and the Asymptotic Bethe Ansatz predictions to these quantities; the result exposes the fact that the dressing phases are different from the original conjectures [7, 23]. A direct Feynman graph calculation [24] of the one-loop worldsheet S matrix in the near-flat space limit of $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$ confirms this conclusion.

These developments underscore the importance of a direct construction of the worldsheet S matrix in these and related integrable worldsheet theories. In $\text{AdS}_5 \times \text{S}^5$ such a construction would further test the crossing equation [19] as well as the assumption that the relevant dressing phase is the minimal one (*i.e.* that no solution of the homogeneous crossing equation needs to be included) [25, 21]. By testing factorization of the S matrix at higher loops and higher points, a direct evaluation of such S-matrix elements would provide a powerful test of integrability and of the existence of an integrability-preserving regularization. Moreover, in $\text{AdS}_5 \times \text{S}^5$, tree-level worldsheet calculations [26] expose the fact that the S-matrix elements depend on the choice of worldsheet (diffeomorphism and κ -symmetry) gauges. By directly constructing the S matrix in a general gauge one may demonstrate explicitly the independence of the target space spectrum on the gauge-choice parameter.

In independent developments, new and powerful methods – the generalized unitarity method [27, 28] and its refinement, the method of maximal cuts [29, 30] and its further generalization to certain massive cases [31, 32] – have been developed for the calculation of scattering amplitudes in various quantum field theories. They have led to the construction with relative ease of a

¹String theory with $M=\text{T}^4$ and no excitations along M may be interpreted as the limit $\alpha \rightarrow 1$ of string theory with $M=\text{S}^3 \times \text{S}^1$. In this limit M decompactifies to $R^3 \times \text{S}^1$; since worldsheet masses depend on α this limit is rather subtle and from the standpoint of the S matrix it involves a nontrivial rearrangement of states. The difference in the topology of M is not observable in the absence of excitations on M.

²String theory in $\text{AdS}_4 \times \text{CP}^3$, dual to the ABJM theory [8], was constructed in supercoset form in [9, 10], finite-gap equations were constructed in [11] and all-loop Asymptotic Bethe Ansatz equations were proposed [12] with specific assumptions on the dressing phase of the S matrix.

whole host of scattering amplitudes in three-, four- and higher-dimensional supersymmetric and non-supersymmetric gauge and gravity theories and to the hope that, in the planar limit, the entire S matrix can be found. The power of the generalized unitarity approach stems from the fact that loop-level amplitudes are constructed in terms of tree-level amplitudes, which depend only on the physical degrees of freedom. They also manifest all the symmetries of the theory, including those that exist only on shell, such as integrability. Quite generally, this method suggests that, up to potential anomalies, the symmetries of the tree-level amplitudes are inherited by loop level amplitudes. The potential anomalies can also be efficiently identified in this approach.

The direct calculation of the tree-level S matrix in $\text{AdS}_5 \times \text{S}^5$ was carried out in [26].³ Direct tree-level calculations have been carried out for a certain massive subsector of string theory in $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$ for general α in [24] and in $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$ with mixed RR and NSNS fluxes in [34].⁴ From a spin chain perspective, the all-loop symmetry-determined parts of the S matrix in these cases were discussed in [17, 16].

Here we will use generalized unitarity to find the two-dimensional cut-constructible part – that is the terms with logarithmic dependence on external momenta – of the one- and two-loop four-point S-matrices for the Green-Schwarz string in $\text{AdS}_5 \times \text{S}^5$, $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$ and $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$ and also to comment on the relation between the $\text{AdS}_4 \times \text{CP}^3$ and $\text{AdS}_5 \times \text{S}^5$ S-matrices. The main ingredients of this construction are the tree-level worldsheet scattering amplitudes. As we shall see, the S matrix can be expected to obey quite generally the factorized structure following from the tree-level symmetry group and integrability. As in all higher-order calculations, regularization is necessary and, ideally, we should use regulated tree amplitudes⁵.

The issue of ultraviolet (UV) regularization of the worldsheet theory in Green-Schwarz form is a thorny one. As a conformal field theory (perhaps with spontaneously broken conformal invariance) the theory is expected to be finite to all orders in perturbation theory; an all-order argument for finiteness however relies on its symmetries, in particular on κ -symmetry. This symmetry is chiral (and has a self-dual parameter) and thus exists only in two dimensions making dimensional regularization unsuitable. This is also related to the presence of the parity-odd Wess-Zumino term in the Green-Schwarz action, which also does not exist⁶ in dimensions other than $d = 2$.

It is not clear what is an example of regularization that preserves all symmetries of the worldsheet theory. In its absence, there are (at least) two possible approaches which, ultimately,

³A direct one-loop calculation was previously attempted unsuccessfully, using worldsheet Feynman graphs, in [33]. The main problem that was encountered was non-cancellation of UV divergences. While such divergences have polynomial momentum dependence and thus can always be eliminated by a local counterterm, their existence goes against the expectation that, at least for supersymmetric ground states, the worldsheet theory does not exhibit infinite renormalization. The surviving UV divergences were also present in the resulting Bethe equations. The reason for the remaining divergences was never clarified. Similar divergences have been reported in Feynman graph calculations in $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$ in [24].

⁴The bosonic S matrix in $\text{AdS}_4 \times \text{CP}^3$ was found in [35].

⁵For example, in dimensional regularization they are the tree-level amplitudes of the d -dimensional theory.

⁶Dimensional regularization can be suitably modified to be applicable in the presence of a Wess-Zumino term. For example, may prescribe some analytic continuation of the two-dimensional Levi-Civita symbol (see e.g. [36]); alternatively, one may do all algebra in two dimensions and continue only the final integrals to $d \neq 2$ (see e.g. [37] for a discussion related to the Green-Schwarz string).

need similar additional input to yield complete S-matrix elements. On general grounds, for any regularization, integrability as well as other classical symmetries broken by the regulator can be restored by the addition to the S matrix of matrix elements of finite local counterterms in the effective action. Their determination relies on the requirement that symmetries be realized at the quantum level. Since these counterterms contribute only rational terms to the S matrix, one may simply determine them from symmetry considerations. Alternatively, two-loop energy calculations [38, 37] suggest that finiteness of the theory is observed if in loop calculations one does all numerator algebra in $d = 2$; this manifestly organizes the result in terms of finite combinations of integrals which may then be evaluated in any (*e.g.* dimensional) regularization. Following this lead we will use two-dimensional tree amplitudes to construct generalized cuts, carry out all numerator algebra in two dimensions and regularize the resulting integrals. While this approach guarantees that the terms exhibiting imaginary parts in $d = 2$ are correctly identified, it leaves open the possibility that terms with no imaginary parts are missed⁷. As in the previous approach, rational terms are determined by symmetry considerations. We will follow this approach here.

It is important to note that the specifics of two-dimensional kinematics allow for the existence of four-point loop integrals with no net momentum flowing through them⁸. One might expect that the interpretation of generalized unitarity as a specific organization of the Feynman diagram expansion will capture at least these (cut-less) terms. Their cuts appear to be singular, however, suggesting that in the absence of a suitable IR regularization their coefficients can at best be determined by a prescription.

This paper is organized as follows. We begin in sec. 2 with a discussion of the structure of the S matrix of a general integrable worldsheet theory with a product-group symmetry and of the relation between the exact (spin chain) and the worldsheet S-matrices. We shall also discuss the structure of the two-dimensional unitarity cuts to all-loop orders, regularization issues, as well as identify the two-dimensional cut-constructible parts of the S matrix. To this end the rational part of the symmetry-determined S matrix will play an important role.

Sec. 3 contains the general form of our results; we collect here in a compact form one- and two-loop amplitudes constructed through the generalized unitarity method and extract their logarithmic dependence on external momenta. We shall illustrate the application of these general expressions to string theory in $\text{AdS}_5 \times \text{S}^5$, $\text{AdS}_4 \times \text{CP}^3$, $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$ and $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$.

In sec. 4 we use generalized unitarity to determine the two-dimensional cut-constructible part of the worldsheet one- and two-loop S matrix in $\text{AdS}_5 \times \text{S}^5$. Our results may be found in eqs. (4.8) and (4.23); we will explicitly demonstrate the exponentiation of the (logarithmic part of the) one-loop dressing phase and thus provide direct evidence for the integrability of the theory at this loop order.

Using the details of the calculations in sec. 4, we briefly comment in sec. 5 on the worldsheet

⁷In particular, in the absence of a complete determination of terms with rational momentum dependence we cannot shed light on the fate of the divergences found in earlier attempts [33]. For the same reason we cannot address the issue pointed out in [16] that the $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$ dressing phases of [22] do not obey the generalized crossing equations constructed there, since the offending terms have a completely rational dependence on external momenta.

⁸In higher dimensions they are regular integrals in the forward limit.

S matrix in $\text{AdS}_4 \times \text{CP}^3$. In particular, we recover the expected result that only a reflectionless S matrix is consistent with worldsheet perturbation theory if the heavy modes of the worldsheet theory are truncated away at the classical level. We also confirm the proposed dressing phase through two-loop order.

In sec. 6 we determine the one-loop worldsheet S matrix in $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$ in the massive sector for general α parameter and confirm the (logarithmic part of the) dressing phases found in [22] – see eqs. (6.16) and (6.19).

In sec. 7 we find the one-loop S matrix in $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$ in the massive sector for a background supported by a mixture of NSNS and RR fluxes parameterized by the q -parameter of [34]; our result may be found in eqs. (7.18) and (7.19). In the $q \rightarrow 0$ limit we find the $m, m' \rightarrow 1$ limit of the $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$ phase of [22]. A further near-flat space limit reproduces the one-loop S-matrix of [24]. In the $q \rightarrow 0$ limit we also construct the two-loop S-matrix and demonstrate the exponentiation of the one-loop dressing phase – see eqs. (7.27) and (7.35).

We close in sec. 8 with remarks on the implications and possible extensions of our results.

The Appendices review the conventions used for the calculation of tree-level worldsheet S-matrices pointing out the differences from the usual four-dimensional ones and collect the details of the exact spin-chain S matrices in the various theories discussed in the paper, the general expression for the two-particle cuts in $\text{AdS}_5 \times \text{S}^5$, one- and two-loop integrals and their (generalized) cuts.

2 Worldsheet perturbation theory for the S matrix

Let us consider a general integrable two-dimensional quantum field theory with a factorized symmetry group $G_1 \otimes G_2$. Examples are the supercoset part of all case for all $\text{AdS}_x \times \text{S}^x \times \text{M}^{10-2x}$ theories, where G_i is either $PSU(2|2)$ or $PSU(1|1)^2$. For such a theory integrability suggests that the four-point S matrix, \mathbb{S} , can be decomposed as

$$\mathbb{S} = \mathbf{S}_{G_1} \otimes \mathbf{S}_{G_2} , \quad (2.1)$$

where \mathbf{S}_{G_1} and \mathbf{S}_{G_2} are S-matrices invariant under the groups G_1 and G_2 , respectively. The excitations scattered by \mathbf{S}_{G_i} are not natural perturbative excitations of the theory; rather, the latter are bilinear⁹ in the former. One may choose to extract an overall phase in eq. (2.1); we will not do this but instead assign all overall phases to the two \mathbf{S}_{G_i} -matrix factors. We shall denote the worldsheet coupling constant by $\hat{g} = \sqrt{\lambda}/(2\pi)$.

It is possible that the symmetry group of the S matrix has further abelian factors, apart from $G_1 \times G_2$, which assign nontrivial charges to representations of $G_1 \times G_2$. An example in this direction is the case for $\text{M}=\text{S}^3 \times \text{S}^1$ where $G_1 = G_2 = PSU(1|1)$ and the additional symmetry is a $U(1)$ factor¹⁰. In such cases integrability no longer requires a factorized \mathbb{S} -matrix of the form (2.1). Since we will reduce the computation of an \mathbb{S} -matrix of the form (2.1) to a separate

⁹While of course physically different, this is formally reminiscent of the KLT relations of standard flat space string theory, in which tree-level supergravity scattering amplitudes are bilinears of tree-level gauge theory scattering amplitudes. The essential difference is that while the latter hold at tree level, wq. (2.1) holds to all orders.

¹⁰We thank B. Hoare for emphasizing this out to us.

computation of the two factors \mathbf{S}_{G_i} , the discussion in this section applies unmodified to the construction of a non-factorized S matrix as well.

2.1 Generalities, parametrization, symmetry restrictions

As usual, the expansion of the worldsheet \mathbb{S} -matrix in the worldsheet coupling constant \hat{g}^{-1} defines the \mathbb{T} matrix

$$\mathbb{S} = \mathbb{1} + \frac{1}{\hat{g}} i\mathbb{T}^{(0)} + \frac{1}{\hat{g}^2} i\mathbb{T}^{(1)} + \mathcal{O}\left(\frac{1}{\hat{g}^3}\right) \equiv \mathbb{1} + i\mathbb{T} , \quad (2.2)$$

which contains all scattering amplitudes. Each factor \mathbf{S}_{G_i} has a similar expansion:

$$\mathbf{S}_{G_i} = \mathbb{1} + \frac{1}{\hat{g}} i\mathbf{T}_{G_i}^{(0)} + \frac{1}{\hat{g}^2} i\mathbf{T}_{G_i}^{(1)} + \mathcal{O}\left(\frac{1}{\hat{g}^3}\right) \equiv \mathbb{1} + i\mathbf{T}_{G_i} . \quad (2.3)$$

We may also formally refer to the entries of $i\mathbf{T}_{G_i}$ as "scattering amplitudes". Integrability of the theory implies that both the $\mathbb{T}^{(0)}$ -matrix and the $\mathbf{T}_{G_i}^{(0)}$ -matrix satisfy the classical limit of the YBE.

The factorization equation (2.1) implies a close relation between the L -loop entries of $i\mathbb{T}$ and the $l \leq L$ -loop entries of $i\mathbf{T}_{G_i}$:

$$i\mathbb{T}^{(L)} = \sum_{l=0}^{L+1} (i\mathbf{T}_{G_1}^{(l-1)}) \otimes (i\mathbf{T}_{G_2}^{(L-l)}) \quad , \quad i\mathbf{T}_{G_i}^{(-1)} = \mathbb{1} . \quad (2.4)$$

Consequently, if integrability is preserved, to find the L -loop \mathbb{T} matrix it suffices to find the L -loop \mathbf{T} matrix, as all the other terms in (2.4) are already determined at lower loops.

If we denote by capital and dotted capital letters the indices acted upon by G_1 and G_2 , respectively, the tree-level factorized \mathbb{T} matrix is given in terms of the \mathbf{T}_{G_i} -matrices as [26]:

$$\mathbb{T}|\Phi_{AA}\Phi'_{BB}\rangle = (-)^{[\dot{A}][B]+[D]}|\Phi_{CA}\Phi'_{DB}\rangle\mathbf{T}_{AB}^{CD} + (-)^{[B][\dot{A}]+[\dot{C}]}|\Phi_{AC}\Phi'_{BD}\rangle\mathbf{T}_{AB}^{\dot{C}\dot{D}} , \quad (2.5)$$

where $[\bullet]$ represents the grade of the argument, which is zero for a bosonic index and unity for a fermionic index. Similarly, the matrix elements of a generic term $\mathbf{T}_i \otimes \mathbf{T}_j \subset \mathbb{T}_{i+j+1}$ between worldsheet states $\Phi_{AA}\Phi'_{BB}$ are related to the matrix elements of \mathbf{T}_i and \mathbf{T}_j between (fictitious) two-particle states $|AB'\rangle$ and $|\dot{A}\dot{B}'\rangle$

$$\mathbf{T}|AB'\rangle = |CD'\rangle(\mathbf{T}_i)_{AB}^{CD} \quad \mathbf{T}_{AB'}^{CD'} = \langle D'C|\mathbf{T}|AB'\rangle \quad (2.6)$$

by

$$\mathbb{T}|\Phi_{AA}\Phi'_{BB}\rangle \supset \mathbf{T}_i \otimes \mathbf{T}_j |\Phi_{AA}\Phi'_{BB}\rangle = (-)^{[\dot{A}][B]+[\dot{C}][D]}|\Phi_{CC}\Phi'_{DD}\rangle(\mathbf{T}_i)_{AB}^{CD}(\mathbf{T}_j)_{AB}^{\dot{C}\dot{D}} ; \quad (2.7)$$

momenta of the primed states are different from momenta of the other states. If either \mathbf{T}_i or \mathbf{T}_j are the identity matrix, *i.e.* if $(\mathbf{T}_i)_{AB}^{CD} = \delta_A^C\delta_B^D$ or $(\mathbf{T}_j)_{AB}^{\dot{C}\dot{D}} = \delta_A^{\dot{C}}\delta_B^{\dot{D}}$, we recover the two terms in (2.5).

Since the world sheet theory is not Lorentz-invariant (by the existence of a fixed vector related to the choice of vacuum), neither the \mathbb{S} -matrix nor the \mathbf{S} -matrix is invariant under crossing transformations

$$\mathbb{S}^{\text{cross}} = \mathcal{C}^{-1} \mathbb{S}^{\text{st}} \mathcal{C} \qquad \mathbf{S}^{\text{cross}} = \mathcal{C}^{-1} \mathbf{S}^{\text{st}} \mathcal{C} . \quad (2.8)$$

Here \mathbb{S}^{st} is the super-transpose of the \mathbb{S} -matrix in the two labels corresponding to the crossed particles,

$$(M^{\text{st}})_{AB} = (-)^{[A][B]+[B]} M_{BA} , \quad (2.9)$$

\mathcal{C} is the charge conjugate matrix and one is also to change the sign of the energy and momentum of the particles that are crossed. The super-transpose is necessary if hermitian conjugation and complex conjugation are defined in the same way as for regular matrices. The standard (relativistic) crossing symmetry is expected to be replaced by the generalized crossing equations suggested in [19], which relate in a nontrivial way \mathbb{S} and $\mathbb{S}^{\text{cross}}$. It is not difficult to see that the two transformations (2.8) are consistent with the factorization (2.1). Through a sequence of crossing transformations (2.8) we shall consistently construct the u -channel cuts by relating them to s -channel without using the explicit form of the crossed \mathbf{S} matrix.

2.2 The perturbative expansion of the worldsheet \mathbf{S} matrix

As mentioned previously, the worldsheet \mathbf{S} matrix is determined by symmetries up to an overall phase denoted by θ_{12} whose general structure in terms of spin-chain variables was discussed in [39]. Its strong coupling expansion at fixed spin-chain momenta is reviewed in Appendix C.

Contact with worldsheet perturbation theory is however made [40, 26] in the strong coupling expansion at fixed worldsheet momenta (the "small momentum expansion")

$$p_{\text{chain}} = \frac{2\pi}{\sqrt{\lambda}} p_{\text{ws}} = \frac{1}{\hat{g}} p_{\text{ws}} . \quad (2.10)$$

In $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$, $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$ and $\text{AdS}_4 \times \text{CP}^3$ the role of the coupling constant is played by a nontrivial function h whose relation to the naive worldsheet coupling \hat{g} in these cases with less-than-maximal supersymmetry is subject to finite renormalization. Taking this limit is straightforward for the state-dependent part of the \mathbf{S} matrices and the result has the same structure as (2.3). We focus here only the features added by the dressing phase to the perturbative expansion of the worldsheet \mathbf{S} matrix. Depending on the theory one may have different dressing phases in different sectors describing the scattering of different multiplets of the symmetry group; the same discussion applies separately for each of them.

The general form of the dressing phase is included in eq. (C.3) with the Zhukowsky variables x^\pm defined in (B.7) and expanded in (D.7). It is not difficult to see that the leading term in the expansion of x^\pm , which is independent of the indices \pm , cancels out in eq. (C.3). It is also not difficult to construct the next order in the expansion of the right-hand side of eq. (C.6) for any $\chi^{(n)}$; thus, in the small momentum expansion, $\theta_{12}^{(n)}$ is $\mathcal{O}(\hat{g}^{-2})$. Extracting the leading \hat{g}^2 factor, $\hat{\theta}_{12}^{(n)} = \hat{g}^2 \theta_{12}^{(n)}$, we have

$$\theta_{12} = \frac{1}{\hat{g}} \sum_{n=0}^{\infty} \frac{1}{\hat{g}^n} \hat{\theta}_{12}^{(n)} , \quad (2.11)$$

with $\hat{\theta}^{(1)}$ contributing at $\mathcal{O}(\hat{g}^{-2})$, *i.e.* at one-loop order. For string theory in $\text{AdS}_5 \times \text{S}^5$, the leading term $\hat{\theta}_{12}^{(0)}$ is also the leading term in the small momentum expansion of the AFS phase [6].

Defining $\hat{\mathbf{S}}$ to be the symmetry-determined part of the S matrix dressed with $\hat{\theta}_{12}^{(0)}$, the \mathbf{S} -matrix can be written as

$$\mathbf{S} = e^{\frac{i}{2}(\theta_{12} - \frac{1}{\hat{g}}\hat{\theta}_{12}^{(0)})} \hat{\mathbf{S}} ; \quad (2.12)$$

Its large \hat{g} expansion identifies the entries of \mathbf{S} containing information on the loop corrections to the dressing phase:

$$\mathbf{S} = \mathbb{1} + \frac{1}{\hat{g}} i \mathbf{T}^{(0)} + \frac{1}{\hat{g}^2} i \left(\hat{\mathbf{T}}^{(1)} + \frac{1}{2} \hat{\theta}_{12}^{(1)} \mathbb{1} \right) + \frac{1}{\hat{g}^3} i \left(\hat{\mathbf{T}}^{(2)} + \frac{i}{2} \hat{\theta}_{12}^{(1)} \mathbf{T}^{(0)} + \frac{1}{2} \hat{\theta}_{12}^{(2)} \mathbb{1} \right) + \mathcal{O} \left(\frac{1}{\hat{g}^4} \right) . \quad (2.13)$$

We see in particular that the one-loop correction to the dressing phase affects only the diagonal entries of the one-loop S matrix.¹¹ Moreover, since the part of the S matrix that is determined by symmetries has rational dependence on momenta and so does the classical phase $\hat{\theta}_{12}^{(0)}$; the only transcendental dependence on external momenta comes from $\theta_{12}^{(i)}$ with $i \geq 1$. A direct demonstration of the structure of the $\mathcal{O}(\hat{g}^{-3})$ term would give strong indication of the exponentiation of the one-loop phase.

For example, for string theory in $\text{AdS}_5 \times \text{S}^5$, the expansion of the first (loop correction to the) dressing phase is [41, 40, 42]

$$\hat{\theta}_{12}^{(1)} = -\frac{1}{\pi} \frac{p^2 p'^2 (\varepsilon \varepsilon' - p p')}{(\varepsilon' p - \varepsilon p')^2} \ln \left| \frac{p_-'}{p_-} \right| + \text{rational} , \quad (2.14)$$

where

$$p_{\pm} = \frac{1}{2}(\varepsilon \pm p) , \quad (2.15)$$

the mass of the worldsheet excitations is set to $m = 1$ and p_+ and p'_+ were eliminated from the argument of the logarithm through the on-shell condition

$$4p_+ p_- = \varepsilon^2 - p^2 = 1 . \quad (2.16)$$

In the sec. 4 we will find this expression for $\hat{\theta}_{12}^{(1)}$ from a direct calculation through the generalized unitarity method; we will also demonstrate that the $\mathcal{O}(\hat{g}^{-3})$ term in the perturbative expansion of the worldsheet S matrix has the form given by (2.13).

2.3 Generalized unitarity and the worldsheet S matrix

The (d -dimensional) generalized unitarity method [27, 28], its implementation in the method of maximal cuts [29, 30] and in cases massive of massive particles [31, 32] provide powerful tools for the construction of one- and higher-loop quantum field theory scattering amplitudes. In the

¹¹We shall use the notation $\mathbf{T}^{(0)}$ rather than $\hat{\mathbf{T}}^{(0)}$ for the leading term in the small momentum expansion of \mathbf{S} because this term is unaffected by the corrections to the dressing phase.

presence of a suitable regularization, such as dimensional regularization, complete amplitudes can be constructed. Terms with rational dependence on momenta (and thus with no cuts) are related to terms that exhibit cuts in the presence of the regulator but disappear as the regulator is removed. We refer to them as "rational terms" or d -dimensional cut constructible terms. The rest, which can be determined from unregularized unitarity cuts, are referred to as two-dimensional cut-constructible terms.

Here we will use the generalized unitarity method to find the two-dimensional cut-constructible parts of the one- and two-loop corrections to the worldsheet S matrix. It is usually the case that rational terms undetermined at some loop level lead to missing terms with logarithmic (or, in general, transcendental) dependence on momenta at the next loop level. While at one loop we do not determine explicitly rational terms, we bypass this issue by making use of the rational terms that are determined by symmetries at one loop; the rational terms proportional to the identity matrix turn out to be irrelevant for finding all logarithms at two loops.¹²

Quite generally, one may use the generalized unitarity method to determine directly the S-matrix elements, which are the scattering amplitudes of the worldsheet theory. The factorization of the \mathbb{T} matrix suggests however that we ought to construct the \mathbf{S} -matrix instead. Indeed, at any loop order L , cuts not already computed at lower loops contribute only to $T_{G_1}^{(L)} \otimes \mathbb{1} + \mathbb{1} \otimes T_{G_2}^{(L)}$. Since the T-matrix elements are substantially simpler than the entries of the \mathbb{T} matrix, we shall focus on the former.

The structure of the tree-level S matrix is tightly constrained by the (assumed) integrability of the theory which, in particular, implies absence of particle production and thus that the number of incoming particles is the same as the number of outgoing ones. Consequently, all components of generalized cuts should also obey this constraint. For example, for the $2 \rightarrow 2$ S matrix at two loops, the three-particle cut with two external particles on each side of the cut vanishes identically. Of course, not all three-particle cuts of two-loop four-point amplitudes vanish identically; an example is the cut with one external line on one side and the other three on the other side of the cut.

At one-loop level, two-particle cuts in two dimensions play a role analogous to that of quadruple cuts in four dimensions. Indeed, since in two dimensions momenta have two components, cutting two internal lines of a one-loop amplitude completely constrains the loop momentum. Thus, two-particle cuts are maximal cuts for one-loop amplitudes; similarly, four-particle cuts are maximal cuts at two loops. Consequently, the coefficients of one-loop bubble integrals are simply given by products of tree-level amplitudes appropriately summed over all possible

¹² Since we do not evaluate directly all rational terms we will not be able to completely address questions regarding the fate of divergences that appeared in previous attempts to compute the one-loop worldsheet S matrix in $\text{AdS}_5 \times \text{S}^5$ [33]. In two dimensions, the only one-loop divergent rational terms are related to tadpole integrals; following the four-dimensional construction of [43], in general theories may be determined from one-particle cuts. As we shall discuss in more detail in sec. 2.4, as a consequence of the factorization of six-point tree-level amplitudes in integrable theories into a product of two four-point amplitudes, single-particle cuts of one-loop four-point amplitudes have an additional hidden cut condition and thus are in fact two-particle cuts. This suggests that there are no tadpole integrals in one-loop amplitudes apart from those related to wave-function renormalization. Since the string tension should not receive infinite renormalization (which is supported by the symmetry-based determination of the terms with completely rational momentum dependence employed here), we may therefore say that, to some extent, generalized unitarity provides a divergence-free construction of the one-loop S matrix.

internal states, in close analogy to the coefficients of box integrals in four-dimensional field theories.

To construct either the \mathbb{T} matrix or, separately, the T_{G_i} -matrix factors it is necessary to have a spectral decomposition of the identity operator in the Hilbert space of states. We will focus here on the two-particle identity operator, for which we can write:

$$\mathbb{1} = |\Phi_{E\dot{E}}\Phi'_{F\dot{F}}\rangle\langle\Phi'^{\dot{F}F}\Phi^{\dot{E}E}| \quad \text{and} \quad \langle\Phi'^{\dot{A}A}\Phi^{\dot{B}B}|\Phi_{E\dot{E}}\Phi'_{F\dot{F}}\rangle = \delta_E^B\delta_{\dot{E}}^{\dot{B}}\delta_F^A\delta_{\dot{F}}^{\dot{A}}. \quad (2.17)$$

Here $|\Phi_{E\dot{E}}\rangle$ corresponds to the state created by the field $\Phi_{E\dot{E}}$ and $\langle\Phi_{E\dot{E}}|$ is the conjugate of that state. We may further split the spectral decomposition (2.17) into $\mathbb{1}_{G_1} \otimes \mathbb{1}_{G_2}$ identity operators:

$$\mathbb{1}_{G_1} \otimes \mathbb{1}_{G_2} = (|EF'\rangle\langle F'E|) \otimes (|\dot{E}\dot{F}'\rangle\langle\dot{F}'\dot{E}|) \quad \text{with scalar product:} \quad \langle A'B|EF'\rangle = \delta_E^B\delta_F^A \quad (2.18)$$

where the primes denote the fact that two excitations carry momenta different from the other two; excitations with different momenta are orthogonal, $\langle A'|E\rangle = 0$.

As usual, we interpret a generalized unitarity cut as selecting from an amplitude the parts that have a certain set of propagators present. Using the spectral decomposition in one of the two group factors and the definition (2.6) of the matrix elements of \mathbb{T} , it is easy to see that the $(L_1 + L_2 + 1) \rightarrow L_1 \times L_2$ s -channel cut of the $(L_1 + L_2 + 1)$ -loop component of the \mathbb{T} matrix is:

$$\begin{aligned} (iT^{(L_1+L_2+1)})_{AB'}^{CD'} \Big|_{s\text{-cut}}^{L_1 \times L_2} &= (i)^2 \langle D'C|(iT^{(L_1)})|EF'\rangle \langle F'E|(iT^{(L_2)})|AB\rangle \\ &= (i)^2 (iT^{(L_1)})_{EF'}^{CD'} (iT^{(L_2)})_{AB'}^{EF'}, \end{aligned} \quad (2.19)$$

where the $(i)^2$ factor originate from the two cut propagators.

To construct the u -channel cut we use the crossing transformation (2.9) to relate it to an s -channel cut. The crossing transformation (2.9) acts consistently on both matrices \mathbb{S} and \mathbf{S} . Here we will use the transformation on the latter one:

$$\begin{aligned} &(-)^{[B][D]+[B]} \langle D'C|(iT^{(L_1+L_2+1)})|AB'\rangle \Big|_{u\text{-cut}}^{L_1 \times L_2} = \langle B'C|(iT^{\text{st}})|AD'\rangle \Big|_{s\text{-cut}}^{L_1 \times L_2} \\ &= (i)^2 \langle B'C|(iT^{(L_1),\text{st}})|EF'\rangle \langle F'E|(iT^{(L_2),\text{st}})|AD'\rangle \\ &= (i)^2 (-)^{[B][F]+[B]} \langle F'C|(iT^{(L_1)})|EB'\rangle (-)^{[F][D]+[F]} \langle D'E|(iT^{(L_2)})|AF'\rangle. \end{aligned} \quad (2.20)$$

In terms of matrix elements this becomes

$$(iT^{(L_1+L_2+1)})_{AB'}^{CD'} \Big|_{u\text{-cut}}^{L_1 \times L_2} = (i)^2 (-)^{([B]+[F])([D]+[F])} (iT^{(L_1)})_{EB'}^{CF'} (iT^{(L_2)})_{AF'}^{ED'}. \quad (2.21)$$

One may also understand the sign factors by *formally* permuting indices and bringing them to the same order as in the s -channel cut. We will use these relations repeatedly in the following sections to construct loop-level worldsheet S -matrix elements in $\text{AdS}_5 \times S^5$, $\text{AdS}_4 \times \text{CP}^3$, $\text{AdS}_3 \times S^3 \times S^3 \times S^1$ and $\text{AdS}_3 \times S^3 \times T^4$.

The t -channel cut is structurally different from the other two: there is no net momentum flow across the cut. Because of this, in the product of two scattering amplitudes in this channel one encounters a kinematic singularity in the form of a factor of $\delta(0)$ or perhaps as the square

of a delta function, *e.g.* $\delta(p_1 - p_3)^2$. This singular momentum configuration is forced on us by two-dimensional kinematics and the integrability of the theory. Clearly, some form of IR regularization is necessary to extract reliable information from this cut.

It is not obvious what such a regularization might be. Continuing external momenta to $d = 2 - 2\epsilon$ would forbid the appearance $\delta(0)$ by allowing *individual* momenta to not be conserved in the scattering process by an $\mathcal{O}(\epsilon)$ amount. As discussed previously however, this regularization cannot preserve all the classical symmetries of the theory. Moreover, since integrability also requires that individual momenta be conserved in a scattering process (for any number of external lines and independently of two-dimensional kinematics), regulating $\delta(0)$ as above would also break integrability. Since the t -channel bubble integral is in fact a constant, we will simply ignore it and determine it together with all the other rational terms from symmetry considerations.

The description of unitarity cuts earlier in this section assumed that scattering amplitudes are normalized in the standard Lorentz-invariant way. As reviewed in Appendix A, worldsheet tree-level S-matrix elements are normalized slightly differently both because of the mode expansion of fields and because the on-shell conditions and momentum conservation are solved using special properties of two-dimensional kinematics. Thus, to apply the usual rules we need to compensate for the solution of the momentum conservation constraint and also adjust the normalization of the creation operators to the relativistic one for each cut leg for both the left and right side of the cut, *i.e.* we need to multiply by $(\sqrt{2\epsilon})^2$ for each cut leg with energy ϵ . For example, for each two-particle cut obtained by multiplying two S-matrix elements in the standard worldsheet normalization we need to supply an additional factor of

$$J = (\sqrt{2\epsilon}\sqrt{2\epsilon'})^2 \left(\frac{d\epsilon}{dp} - \frac{d\epsilon'}{dp'} \right) ; \quad (2.22)$$

For a standard dispersion relation this is:

$$(\sqrt{2\epsilon}\sqrt{2\epsilon'})^2 \frac{\epsilon' p - \epsilon p'}{\epsilon \epsilon'} = 4(\epsilon' p - \epsilon p') = \frac{2(m^2 p'^2 - m'^2 p^2)}{p_- p'_-} . \quad (2.23)$$

We shall denote this expression by $J_{s,u}$, depending on the channel in which it appears. It is not difficult to see however that $J_s = J_u$ so we will at times also simply denote it by J .

2.4 On higher loops, regularization, factorization and related issues

Two-particle cuts are sufficient for one-loop calculations. In general, the 2-, 3-, \dots , $2L$ -particle generalized cuts of an L -loop amplitude determine its cut-constructible part. It is interesting to examine the cuts needed for higher-loop calculations and understand their structure in view of the special properties of the theory.

Let us assume that we focus on the massive sector of the theory and that we choose a regularization that preserves the integrability; for example, a possibility is to assume that, since the theory should be finite both in the IR (being massive) and in the UV (being a theory with spontaneously broken conformal invariance) no regularization is necessary. A regularization preserving the integrability of the theory will also preserve the absence of particle production.

In particular, all nonzero amplitudes have an even number of external legs and an equal number of incoming and outgoing particles; $2k$ -point amplitudes are a sequence of four-point scattering events. This structure places a number of constraints on the cuts that determine the higher-loop S-matrix elements.

We first notice that any cut that is a product of tree-level amplitudes is also a maximal cut. Indeed, in an integrable quantum field theory the higher-point S-matrix elements are given by the sum of products of four-point S-matrix elements; the internal lines connecting them are on shell and, as such, may be interpreted as cut propagators. Thus, a generalized cut of an L -loop amplitude that is a product of tree-level amplitudes is also naturally the cut of the same L -loop amplitude in which all tree-level factors are four-point tree-level amplitudes – *i.e.* the maximal cut of the amplitude. The coefficient of an L -loop integral that has a non-zero maximal cut is simply given – for any L – by the product of tree-level amplitudes appropriately summed over all the possible states crossing the generalized cut. For example, the two-loop six-point tree-level amplitude the two-loop factorizes as a sum of products of two four-point amplitudes; Then, the three-particle cut with one external leg on one side and three on the other side of the cut may be interpreted as contains a hidden on-shell condition for a propagator internal to the six-point tree-level amplitude [26] and thus it is in fact a four-particle cut.

In the simplest choice of regularization described above such cuts do not determine all terms with logarithmic dependence beyond one loop. For this purpose, and to be able to include the contribution of symmetry-determined rational terms at lower-loop orders, it is useful to consider cuts that break up an amplitude into a product of an L_1 -loop and an L_2 -loop amplitude. Integrability then implies that each such amplitudes is a sum of products of $l_i \leq L_1$ and $l'_j \leq L_2$ four-point amplitudes with $\sum l_i = L_1$ and $\sum l'_j = L_2$. In non-integrable theories, focusing on a single term in this sum requires imposing additional cut conditions. In contrast, in an integrable theory these conditions are naturally present and do not impose new restrictions on the amplitudes building up the generalized cut. In each amplitude factor we may recursively include the symmetry-determined rational terms at l_i -loop order. Based on the structure of one-loop integrals and on the structure of two-particle cuts it is possible to argue¹³ quite generally that $(L-1)$ -loop rational terms proportional to the identity matrix in field space – and thus undetermined by symmetries – do not contribute to L -loop terms with logarithmic dependence on momenta. One may check on a case by case basis whether rational terms which are not determined by symmetries at $l \leq (L-2)$ -loop order make any contributions to the logarithmic terms at L -loops. The argument here guarantees that no such contributions exist at two loops.

An argument similar in spirit to the one in the previous paragraph suggests that logarithmic terms at L -loops preserve the factorization (2.4) of the \mathbb{T} matrix. Indeed, let us consider the cut in which one side of the cut is a four-point tree-level amplitude. Since

$$i\mathbb{T}^{(0)} = (iT_{G_1}^{(0)}) \otimes \mathbb{1} + \mathbb{1} \otimes (iT_{G_1}^{(0)}) , \quad (2.24)$$

it follows that the product $(i\mathbb{T}^{(L)})(iT^{(0)})$ will exhibit the factorization (2.4) after the cut conditions are released and the integrals are evaluated. This argument is sufficient to guarantee

¹³A property of the two-dimensional one-loop bubble integrals \tilde{I}_s and \tilde{I}_u listed in Appendix F, is that only their difference contains logarithms; thus, the difference of the coefficients of these integrals determines the coefficient of the logarithm of external momenta.

factorization through two-loop order, where only cuts of the type above are important. While at higher loops the factorization (2.4) clearly holds at the level of cuts, it is less straightforward to see it in general once cut conditions are removed. Nevertheless, consistency of cuts in all channels will relate all contributions of $\mathbb{T}^{(L)}$ to the ones that are factorized following the argument above, suggesting that it is plausible that eq. (2.4) indeed holds to all orders in perturbation theory.

A class of theories that feature in the context of the gauge/string duality are those containing multiplets of the symmetry group that have different masses. An interesting question which arose in the comparison of worldsheet and spin chain S matrices is whether calculations in the theory truncated to some subset of fields (*e.g.* all fields with some subset of masses) yield the same result as calculations in the complete theory. From a Lagrangian point of view this is clearly impossible unless the desired subset is decoupled from the other fields. Generalized unitarity provides more structure since it makes use of properties of the S matrix not immediately visible in the Lagrangian. To understand the terms that are missed by restricting ourselves to a subset of fields, let us start with an integrable quantum field theory with particles of different masses and ignore one of them, denoted by φ . We aim to construct the scattering amplitudes of the remaining fields from their generalized cuts; ignoring φ means that in generalized unitarity cuts we sum over all states except φ . It is not difficult to see that, in a generic quantum field theory, these steps result in an incorrect S matrix.

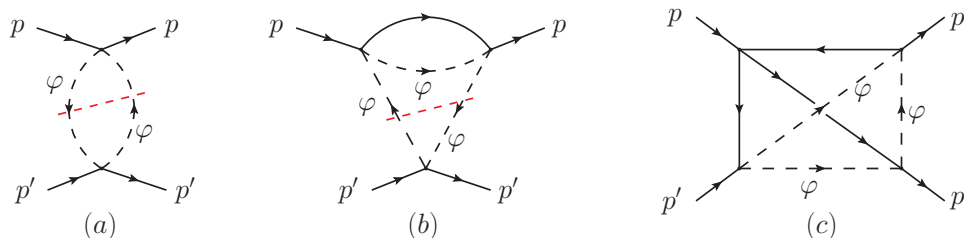


Figure 1: Integrals with fields that are truncated away at the classical level. The external momentum configuration guarantees that in (a) and (b) there are φ states crossing the cut. The one- and two-loop integrals are constants, independent of external momenta. The three-loop integral depends on both external momenta and therefore need not be a rational function.

If the tree-level S matrix has further special properties, then the two-dimensional cut-constructible terms may be reproduced correctly. Indeed, let us assume that masses of individual particles are conserved and that the scattering of φ (or, more generally, the scattering of fields we want to truncate away) is *reflectionless*, *i.e.* that there is no momentum exchanged between the two scattered particles. All theories with different worldsheet masses that we will discuss in later sections are of this type¹⁴. With such assumptions we see that there is no net momentum flow across any cut putting on shell only φ fields, such as the cuts in fig. 1(a) and 1(b). Indeed, since integrability implies that momenta of φ fields are separately conserved, φ

¹⁴This includes the massless particles that do not decouple from the supercoset sigma model for strings in $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$ but can be consistently truncated away at the classical level [34].

fields run in the loop only if the external momenta are chosen as shown. The two integrals are momentum-independent because the integrand depends on at most one external momentum and thus disappears upon use of the on-shell condition. It is not difficult to argue in a similar way that all integrals having this property – that they have no momentum flow across some generalized cut that sets on shell only φ fields – are independent of external momenta and thus contribute only to the rational part of the S matrix.

This argument guarantees that, at one and two loops, all logarithmic terms are correctly reproduced by calculations in the theory obtained by truncating away fields that scatter reflectionlessly off the remaining ones. At three loops and beyond it is possible to construct integrals – such as the one in fig. 1(c) – that do not have any generalized cut crossed only by φ -type fields and thus the argument here does not immediately imply that such integrals are rational functions. Thus, at three loops and beyond quantum calculations in the truncated theory do not necessarily yield the complete result.

In a theory exhibiting nontrivial two-point functions the construction of the S matrix through the LSZ reduction requires that the physical pole of the two-point function of fields be identified and that its residue be correctly included in the reduction of Green's functions to S-matrix elements. In general, this implies that the naive L -loop amplitudes are corrected by the addition of lower-loop amplitudes multiplied by the residue of the two-point function. In all theories we will discuss in later sections it is expected that the first correction to the dispersion relation (and hence to the two-point function) is at two loops (see *e.g.* [44] for a calculation in the near-flat space limit of $\text{AdS}_5 \times \text{S}^5$). Thus, through two loops, only the rational terms in the S matrix will be affected; since we determine the rational terms from symmetry considerations we will ignore the corrections to the propagator. At higher loops it is, of course, important to include such contributions.

One way to construct scattering amplitudes through the generalized unitary method is to begin with an ansatz for amplitudes in terms of Feynman-like integrals whose coefficients are subsequently determined by comparing the generalized cuts of the ansatz with the generalized cuts of amplitudes constructed in terms of lower-order amplitudes. The ansatz is based on the structure of the Feynman graphs of the theory. It is interesting to note that, in all cases we are interested in, all cuts that break up an L -loop amplitude into a product of tree-level amplitudes completely freeze the loop momenta and therefore cannot distinguish between scalar integrals and tensor integrals. Since rational terms are supplied separately, one may thus attempt to construct ansätze in terms of only scalar integrals; such ansätze turn out to be sufficient at one and two loops.

3 General expressions for one- and two-loop amplitudes

Due to the special properties of S-matrices of the Green-Schwarz string in $\text{AdS}_5 \times \text{S}^5$, $\text{AdS}_4 \times \text{CP}^3$, $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$ and $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$, generalized unitarity allows us to derive compact general expressions for all logarithmic terms of the one- and two-loop amplitudes in these theories. We shall outline the derivation in this section; we will then proceed in later sections to discuss each of these theories separately, pointing out the features specific to each of them.

For a compact notation we will interpret all indices used in sec. 2 as multi-indices while

keeping their capital letter appellation, A, B, C, \dots . Each multi-index stands for the set (field label, mass of field, sector).¹⁵ Not all entries are relevant in all theories. For example, in $\text{AdS}_5 \times \text{S}^5$ all fields have the same mass and there are no left and right excitations; thus, in this case only the field label is relevant. In $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$ however all entries are important. The grade $[A]$ of a multi-index is the grade of the field label (0/1 for bosons/fermions).

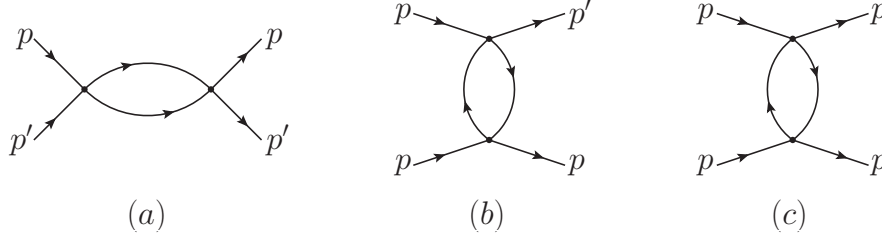


Figure 2: The integrals appearing in the one-loop four-point amplitudes. Tensor integrals can be reduced to them as well as to tadpole integrals, which are momentum-independent.

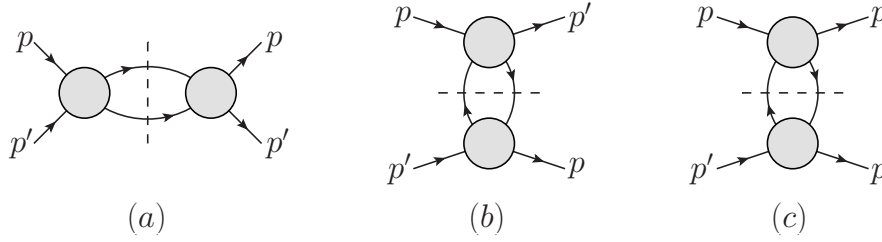


Figure 3: Two-particle cuts of the one-loop four-point amplitudes

3.1 The general expression for one-loop amplitudes

The integrals that can appear in one-loop amplitudes are shown in fig. 2; their details depend on the worldsheet spectrum. The general structure of one-loop amplitudes is

$$iT^{(1)} = \frac{1}{2}C_s \tilde{I}_s + \frac{1}{2}C_u \tilde{I}_u + \frac{1}{2}C_t \tilde{I}_t + \text{rational} . \quad (3.1)$$

¹⁵ The last entry of the multi-index refers to transformation of the excitation under some representation (L) or the conjugate representation (R) of the symmetry group, such as $PSU(1|1)^2$ for strings in $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$. They are also referred to as left- and right-moving excitations in [23]. In integrable models, the concept of left- and right-excitations was originally introduced in the context of massless factorized scattering [45]. In that case the LL and RR scattering is non-perturbative and LR and RL is perturbative. While we use the original notation, perhaps a more natural one is $L \leftrightarrow +$, $R \leftrightarrow -$ used in [34] in a related context. For us all states are massive and therefore a left-motion can be transformed into right-motion by a change of two-dimensional frame. Because of this scattering in all four sectors is perturbative. The notation RR (sector of the S matrix) should not be confused with that for the RR flux.

with the factors of $1/2$ being the symmetry factors of bubble integrals; the expression for these integrals \tilde{I} as well as their equal-mass versions are collected in Appendix F. The coefficients $C_{u,s}$ are tensors in field space. As discussed in the previous section, we cannot reliably determine the coefficient C_t due to the kinematic singularity of the t -channel cut. As emphasized there, at one-loop, two-particle cuts are also maximal; this simply implies that the two coefficients, C_s and C_u , have simple expressions in terms of the tree-level S matrix $iT^{(0)}$:

$$\begin{aligned} (C_s)_{AB'}^{CD'} &= (i)^2 J_s \sum_{E,F'} (iT^{(0)})_{EF'}^{CD'} (iT^{(0)})_{AB'}^{EF'} \\ (C_u)_{AB'}^{CD'} &= (i)^2 J_u \sum_{E,F'} (-)^{([B]+[F])([D]+[F])} (iT^{(0)})_{EB'}^{CF'} (iT^{(0)})_{AF'}^{ED'} . \end{aligned} \quad (3.2)$$

The Jacobians J_s and J_u are the adjustment factors eq. (2.23) needed to transform the S-matrix elements to the relativistic normalization. We recall that the precise form of these factors depends on the dispersion relation.

The explicit form of the one-loop bubble integrals from Appendix F implies that the difference

$$\frac{C_s}{J_s} - \frac{C_u}{J_u} \quad (3.3)$$

governs the logarithmic dependence of one-loop amplitudes on external momenta. Noticing that only \tilde{I}_s has a rational component, we can cleanly separate all logarithmic dependence on momenta by organizing $iT^{(1)}$ as

$$iT^{(1)} = \frac{1}{2} \frac{C_s}{J_s} (J_s \tilde{I}_s + 1) + \frac{1}{2} C_u \tilde{I}_u + i\tilde{T}^{(1)} , \quad (3.4)$$

where

$$i\tilde{T}^{(1)} = i\hat{T}^{(1)} + i\Phi \mathbb{1} . \quad (3.5)$$

$i\hat{T}^{(1)}$ was introduced in eq. (2.13) as the $\mathcal{O}(\hat{g}^{-2})$ term in the small momentum expansion of the symmetry-determined part of the S matrix dressed with the classical part of the dressing phase and Φ is the contribution of rational terms in the dressing phase which are not determined by symmetries. We will notice that, in all theories we analyze, the off-diagonal entries of $i\hat{T}^{(1)}$ are proportional to the corresponding tree-level amplitudes.

3.2 The general expression for two-loop amplitudes

To construct the two-loop correction to the worldsheet S matrix, we begin by constructing an ansatz that contains all two-loop integrals that can appear and have a logarithmic dependence on momenta. We list in fig. 4 all integrals that have maximal cuts that are kinematically non-singular. The cuts determining their coefficients are shown in fig. 5. In addition, we will include integrals that do not have non-singular maximal cuts but have single two-particle cuts; they are just products of the s - or u -channel bubble integrals with the t -channel integral \tilde{I}_t . The

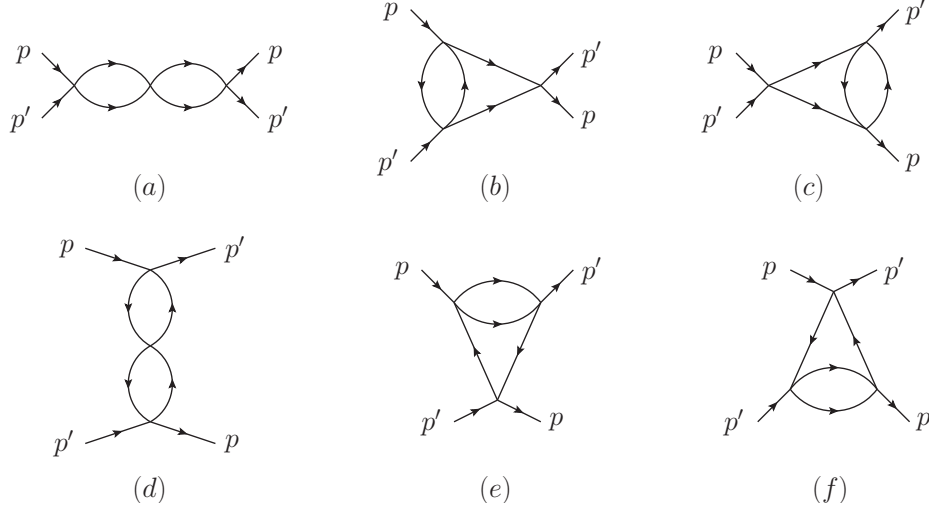


Figure 4: The integrals appearing in the two-loop four-point amplitudes. Each cut in fig. 5 determines the coefficient of one of these integrals. There exist, of course, other two-loop four-point integrals; the structure of the Lagrangian suggests that integrals with vertices with an odd number of edges cannot appear while the integral with a six-point vertex is momentum-independent and thus it can contribute only to terms with rational momentum dependence.

latter factor is constant and can thus be absorbed in the coefficient of the one-loop integrals. The ansatz is therefore:

$$\begin{aligned}
iT^{(2)} &= \frac{1}{4}C_a\tilde{I}_a + \frac{1}{2}C_b\tilde{I}_b + \frac{1}{2}C_c\tilde{I}_c + \frac{1}{4}C_d\tilde{I}_d + \frac{1}{2}C_e\tilde{I}_e + \frac{1}{2}C_f\tilde{I}_f \\
&+ \frac{1}{2}C_{s,\text{extra}}\tilde{I}_s + \frac{1}{2}C_{u,\text{extra}}\tilde{I}_u \\
&+ \text{rational} ,
\end{aligned} \tag{3.6}$$

where we explicitly included the symmetry factors of integrals. A sum over possible distributions of internal masses is assumed. While here we are keeping the setup general, in all our explicit two-loop calculations we shall have all masses equal. The relevant integrals are listed in Appendix F. This ansatz manifestly satisfies the vanishing of the three-particle cut containing five-point tree-level amplitudes.

As at one loop, the coefficients C are tensors in field space. Maximal cuts determine them in terms of the tree-level S-matrix elements or, alternatively, in terms of the tree-level S-matrix elements and one-loop integral coefficients (3.2):

$$\begin{aligned}
(C_a)_{AB'}^{CD'} &= (i)^2 J_s \sum_{G,H'} (iT^{(0)})_{GH'}^{CD'} (C_s)_{AB'}^{GH'} \\
(C_b)_{AB'}^{CD'} &= (i)^2 J_s \sum_{G,H'} (iT^{(0)})_{GH'}^{CD'} (C_u)_{AB'}^{GH'} \\
(C_c)_{AB'}^{CD'} &= (i)^2 J_s \sum_{G,H'} (C_s)_{GH'}^{CD'} (iT^{(0)})_{AB'}^{GH'}
\end{aligned}$$

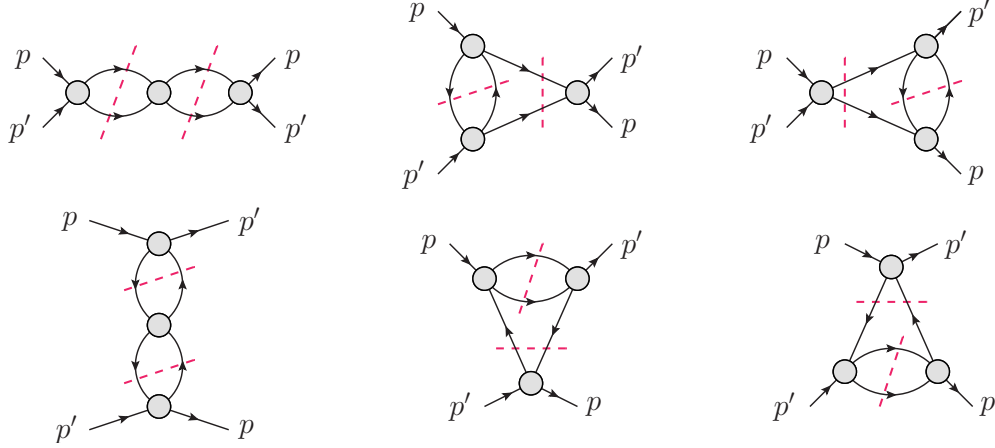


Figure 5: Iterated two-particle cuts of two-loop four-point amplitudes. They are all maximal cuts (in two dimensions). It is not possible to relax the cut condition on any propagator either because the corresponding tree-level amplitude does not exist or because the resulting higher-point tree amplitude has an on-shell propagator as a consequence of integrability and S-matrix factorization. As discussed in sec. 2 all cuts of a four-point two-loop amplitude which is a product of tree amplitudes is equivalent to a sum of the cuts shown here.

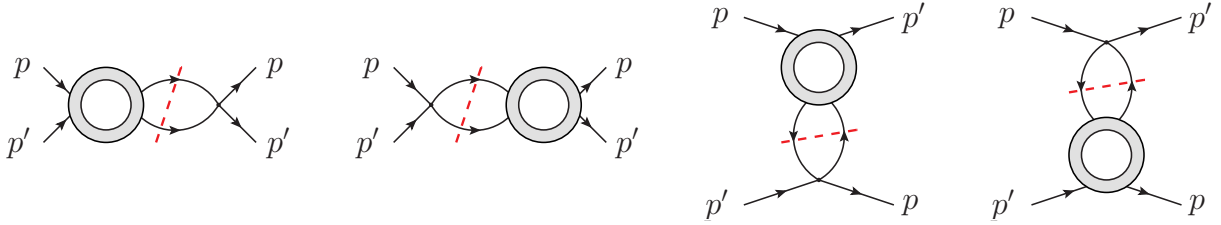


Figure 6: The single two-particle cuts of two-loop four-point amplitudes. They are used to determine the subleading logarithms not captured by maximal cuts.

$$\begin{aligned}
(C_d)^{CD'}_{AB'} &= (i)^2 J_u \sum_{G,H'} (-)^{([B]+[H])([D]+[H])} (iT^{(0)})^{CH'}_{GB'} (C_u)^{GD'}_{AH'} \\
(C_e)^{CD'}_{AB'} &= (i)^2 J_u \sum_{G,H'} (-)^{([B]+[H])([D]+[H])} (iT^{(0)})^{CH'}_{GB'} (C_s)^{GD'}_{AH'} \\
(C_f)^{CD'}_{AB'} &= (i)^2 J_u \sum_{G,H'} (-)^{([B]+[H])([D]+[H])} (C_s)^{CH'}_{GB'} (iT^{(0)})^{GD'}_{AH'} .
\end{aligned} \tag{3.7}$$

With these coefficients the three-particle cuts of two-loop amplitudes are also reproduced.

The two remaining coefficients, $C_{s,\text{extra}}$ and $C_{u,\text{extra}}$, are found by comparing the single two-particle cuts of the ansatz with the single two-particle cut of the two-loop amplitude (see fig. 6).

The former are given in terms of complete one-loop amplitudes $iT^{(1)}$

$$iT^{(2)CD'}_{AB'} \Big|_{s\text{-cut}} = (i)^2 J_s \sum_{G,H'} \left((iT^{(0)})^{CD'}_{GH'} (iT^{(1)})^{GH'}_{AB'} + (iT^{(1)})^{CD'}_{GH'} (iT^{(0)})^{GH'}_{AB'} \right) \quad (3.8)$$

$$iT^{(2)CD'}_{AB'} \Big|_{s\text{-cut}} = (i)^2 J_u \sum_{G,H'} (-)^{([B]+[H])([D]+[H])} \left((iT^{(0)})^{CH'}_{GB'} (iT^{(1)})^{GD'}_{AH'} + (iT^{(1)})^{CH'}_{GB'} (iT^{(0)})^{GD'}_{AH'} \right) \quad (3.9)$$

while the latter are given by

$$iT^{(2)} \Big|_{s\text{-cut}}^{\text{eq. 3.6}} = \frac{C_a}{J_s^2} ((J_s \tilde{I}_s + 1) - 1) + \frac{1}{2} \left(\frac{C_b}{J_s J_u} + \frac{C_c}{J_u J_s} \right) J_u \tilde{I}_u + \frac{1}{2} \left(\frac{C_b}{J_s} + \frac{C_c}{J_s} \right) \tilde{I}_t + \frac{C_{s,\text{extra}}}{J_s} , \quad (3.10)$$

$$iT^{(2)} \Big|_{u\text{-cut}}^{\text{eq. 3.6}} = \frac{C_d}{J_u^2} J_u \tilde{I}_u + \frac{1}{2} \left(\frac{C_e}{J_s J_u} + \frac{C_f}{J_u J_s} \right) ((J_s \tilde{I}_s + 1) - 1) + \frac{1}{2} \left(\frac{C_e}{J_s} + \frac{C_f}{J_s} \right) \tilde{I}_t + \frac{C_{u,\text{extra}}}{J_u} . \quad (3.11)$$

Using eq. (3.7) it is not difficult to see that the coefficients of $(J_s \tilde{I}_s + 1)$ and $J_u \tilde{I}_u$ are the same in eqs. (3.8) and (3.10) as well as in eqs. (3.9) and (3.11). This is a consequence of identities between the one-loop and the two-loop maximal cuts stemming from the fact that both of them are given as sums of products of tree-level amplitudes; it is also a manifestation of the consistency of the generalized unitarity method.

We can therefore immediately read off the remaining coefficients; from eqs. (3.8) and (3.9) only the terms in which $T^{(1)}$ is replaced with its rational part $\tilde{T}^{(1)}$ contribute:

$$\begin{aligned} \frac{1}{J_s} (C_{s,\text{extra}})_{AB'}^{CD'} &= (i)^2 J_s \sum_{G,H'} \left((iT^{(0)})^{CD'}_{GH'} (i\tilde{T}^{(1)})^{GH'}_{AB'} + (i\tilde{T}^{(1)})^{CD'}_{GH'} (iT^{(0)})^{GH'}_{AB'} \right) \\ &\quad + \frac{(C_a)_{AB'}^{CD'}}{J_s^2} - \frac{1}{2} \left(\frac{(C_e)_{AB'}^{CD'}}{J_s} + \frac{(C_f)_{AB'}^{CD'}}{J_s} \right) I_t , \end{aligned} \quad (3.12)$$

$$\begin{aligned} \frac{1}{J_u} (C_{u,\text{extra}})_{AB'}^{CD'} &= (i)^2 J_u \sum_{G,H'} (-)^{([B]+[H])([D]+[H])} \left((iT^{(0)})^{CH'}_{GB'} (i\tilde{T}^{(1)})^{GD'}_{AH'} + (i\tilde{T}^{(1)})^{CH'}_{GB'} (iT^{(0)})^{GD'}_{AH'} \right) \\ &\quad + \frac{1}{2} \left(\frac{(C_e)_{AB'}^{CD'}}{J_s J_u} + \frac{(C_f)_{AB'}^{CD'}}{J_u J_s} \right) - \frac{1}{2} \left(\frac{(C_e)_{AB'}^{CD'}}{J_s} + \frac{(C_f)_{AB'}^{CD'}}{J_s} \right) I_t . \end{aligned} \quad (3.13)$$

The structure of one-loop integrals implies again that any logarithmic dependence on momenta that is not fixed by two-loop maximal cuts depends only on the difference

$$\frac{C_{s,\text{extra}}}{J_s} - \frac{C_{u,\text{extra}}}{J_u} . \quad (3.14)$$

One can check that all terms proportional to the identity matrix in $\tilde{T}^{(1)}$, $\tilde{T}^{(1)CD'}_{AB'} \propto \delta_A^C \delta_{B'}^{D'}$, cancel out in this difference and thus all two-loop logarithmic terms can be determined unambiguously. If desired (*e.g.* for a three-loop calculation), the two-loop rational terms $i\hat{T}^{(2)}$ can be supplied separately, by expanding the symmetry-determined part of the S matrix.

Using the explicit expressions for the integrals in Appendix F one can see that in theories in which all worldsheet masses are equal the coefficient of the double logarithm is given by the combination

$$\mathcal{C}_{\ln^2} = \frac{1}{8\pi^2 J^2} (-2C_a + C_b + C_c - 2C_d + C_e + C_f) , \quad (3.15)$$

while the coefficient of the simple logarithm $\ln \frac{p'_-}{p_-}$ is given by

$$\begin{aligned} \mathcal{C}_{\ln^1} = & \frac{i}{2\pi} \left[\frac{1}{2J^2} (2C_a - C_b - C_c) \right. \\ & \left. - \frac{1}{J} (C_{s,\text{extra}} - C_{u,\text{extra}}) - \frac{i}{8\pi J} (C_b + C_c - C_e - C_f) \right] . \end{aligned} \quad (3.16)$$

Here we used the fact that $J_s = J_u$.

In the following sections we shall compute the one-loop integral coefficients for $\text{AdS}_5 \times \text{S}^5$, $\text{AdS}_4 \times \text{CP}^3$ and $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$ with NSNS and RR fluxes. We shall find that the difference

$$\frac{C_s}{J_s} - \frac{C_u}{J_u} \quad (3.17)$$

is proportional to the identity matrix in each of the different sectors of the S matrix as labeled by the third entry of the multi-index (*e.g.* LL and LR scattering, see footnote 15); the proportionality coefficient can be identified with the one-loop dressing phase in each sector, cf. eq. (2.13). We shall compute the two-loop integral coefficients for $\text{AdS}_5 \times \text{S}^5$, $\text{AdS}_4 \times \text{CP}^3$ and $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$ with RR flux and find that

$$\begin{aligned} \mathcal{C}_{\ln^2} &= 0 \\ \mathcal{C}_{\ln^1} &= \frac{i}{4\pi^2 J^2} (2C_a - C_b - C_c) \propto -\frac{1}{2} T^{(0)} . \end{aligned} \quad (3.18)$$

The proportionality coefficient in each sector is given by the coefficient of the identity matrix in $C_s/J_s - C_u/J_u$. This demonstrates the exponentiation of the one-loop dressing phase in each sector and thus provides support for two-loop integrability in all sectors.

4 The S matrix for strings in $\text{AdS}_5 \times \text{S}^5$

In this section we will use generalized unitarity and the special properties of two-dimensional integrable quantum field theories discussed in sec. 2 to recover the known S matrix for string theory in $\text{AdS}_5 \times \text{S}^5$. In this case $G_1 = G_2 = PSU(2|2)$. The worldsheet theory contains eight bosons and eight fermions of equal mass which we shall normalize to $m = 1$. Therefore, the multi-indices relabeling the S-matrix have a single entry – the field label, which is just the fundamental representation of $PSU(2|2)$. Each of them is represented by a pair of a two-component bosonic index and a two-component fermionic index, such as (a, α) *etc.*, each acted upon by an $SU(2) \subset PSU(2|2)$. The symmetry-determined part of the S matrix was found

in [46] and the dressing phase conjectured in [25] was tested through two loops in the $SL(2)$ sector. The T matrix is parametrized as

$$\begin{aligned} T_{ab}^{cd} &= A \delta_a^c \delta_b^d + B \delta_a^d \delta_b^c, & T_{ab}^{\gamma\delta} &= C \epsilon_{ab} \epsilon^{\gamma\delta}, \\ T_{\alpha\beta}^{\gamma\delta} &= D \delta_\alpha^\gamma \delta_\beta^\delta + E \delta_\alpha^\delta \delta_\beta^\gamma, & T_{\alpha\beta}^{cd} &= F \epsilon_{\alpha\beta} \epsilon^{cd}, \\ T_{a\beta}^{c\delta} &= G \delta_a^c \delta_\beta^\delta, & T_{\alpha b}^{\gamma d} &= L \delta_\alpha^\gamma \delta_b^d, \\ T_{a\beta}^{\gamma d} &= H \delta_a^d \delta_\beta^\gamma, & T_{\alpha b}^{c\delta} &= K \delta_\alpha^\delta \delta_b^c. \end{aligned} \quad (4.1)$$

Each of the coefficients A...K has an inverse- \hat{g} expansion, *e.g.*

$$A = \frac{1}{\hat{g}} A^{(0)} + \frac{1}{\hat{g}^2} A^{(1)} + \dots, \quad (4.2)$$

and similarly for all other coefficients. Here $A^{(0)}$, *etc.* are tree-level S-matrix elements, *i.e.* the entries of $T^{(0)}$ introduced in eq. (2.13).

The action has standard Lorentz-invariant quadratic terms but interactions break this symmetry. The tree-level S-matrix elements were found in [26]:

$$\begin{aligned} A^{(0)}(p, p') &= \frac{1}{4} \left[(1 - 2a) (\varepsilon' p - \varepsilon p') + \frac{(p - p')^2}{\varepsilon' p - \varepsilon p'} \right], \\ B^{(0)}(p, p') &= -E^{(0)}(p, p') = \frac{pp'}{\varepsilon' p - \varepsilon p'}, \\ C^{(0)}(p, p') &= F^{(0)}(p, p') = \frac{1}{2} \frac{\sqrt{(\varepsilon + 1)(\varepsilon' + 1)} (\varepsilon' p - \varepsilon p' + p' - p)}{\varepsilon' p - \varepsilon p'}, \\ D^{(0)}(p, p') &= \frac{1}{4} \left[(1 - 2a) (\varepsilon' p - \varepsilon p') - \frac{(p - p')^2}{\varepsilon' p - \varepsilon p'} \right], \\ G^{(0)}(p, p') &= -L^{(0)}(p', p) = \frac{1}{4} \left[(1 - 2a) (\varepsilon' p - \varepsilon p') - \frac{p^2 - p'^2}{\varepsilon' p - \varepsilon p'} \right], \\ H^{(0)}(p, p') &= K^{(0)}(p, p') = \frac{1}{2} \frac{pp'}{\varepsilon' p - \varepsilon p'} \frac{(\varepsilon + 1)(\varepsilon' + 1) - pp'}{\sqrt{(\varepsilon + 1)(\varepsilon' + 1)}}. \end{aligned} \quad (4.3)$$

Here $\varepsilon = \sqrt{1 + p^2}$ denotes the relativistic energy. The parameter a reflects the dependence of the S matrix on the choice of physical states selected by the gauge-fixing of two-dimensional diffeomorphism invariance. We shall see that it does not affect the logarithmic part of the dressing phase.

As noted in [26], the tree-level S matrix determined by the coefficients (4.3) differs from the one obtained by expanding the one in [46] by terms linear in the particle's momenta. These terms may be accounted for by a suitable rephasing of the S matrix in [46] (included for convenience in Appendix B) as

$$\begin{aligned} \hat{\mathbf{A}}^B &= \mathbf{A}^B e^{i(1-2a)(p-p')} & \hat{\mathbf{B}}^B &= \mathbf{B}^B e^{i(1-2a)(p-p')} \\ \hat{\mathbf{C}}^B &= \mathbf{C}^B e^{i((\frac{5}{4}+b-2a)p - (\frac{1}{4}-b-2a)p')} & \hat{\mathbf{D}}^B &= \mathbf{D}^B e^{i((\frac{1}{2}-2a)p - (\frac{1}{2}-2a)p')} \\ \hat{\mathbf{E}}^B &= \mathbf{E}^B e^{i((\frac{1}{2}-2a)p - (\frac{1}{2}-2a)p')} & \hat{\mathbf{F}}^B &= \mathbf{F}^B e^{i((\frac{1}{4}-b-2a)p - (\frac{5}{4}+b-2a)p')} \\ \hat{\mathbf{G}}^B &= \mathbf{G}^B e^{i(-\frac{1}{2}p + (1-2a)(p-p'))} & \hat{\mathbf{H}}^B &= \mathbf{H}^B e^{i((\frac{3}{4}+b-2a)p - (\frac{3}{4}+b-2a)p')} \\ \hat{\mathbf{K}}^B &= \mathbf{K}^B e^{i((\frac{3}{4}-b-2a)p - (\frac{3}{4}-b-2a)p')} & \hat{\mathbf{L}}^B &= \mathbf{L}^B e^{i(\frac{1}{2}p' + (1-2a)(p-p'))}. \end{aligned} \quad (4.4)$$

One may check that the \mathbf{S} -matrix with these coefficients obeys the graded untwisted Yang-Baxter equation; it is therefore a particular case of the \mathbf{S} -matrix constructed in [47]. The phases added to $\mathbf{A}^B, \mathbf{B}^B, \mathbf{D}^B, \mathbf{E}^B, \mathbf{G}^B, \mathbf{L}^B$ eliminate the terms linear in momenta that are different between the spin chain and the world sheet calculations; the other phases, depending on the free parameter b may be adjusted (or eliminated) by a rephasing of external states.

4.1 The logarithmic terms of the one-loop $\text{AdS}_5 \times \text{S}^5$ \mathbf{S} matrix

The one-loop amplitudes have the general form (3.1) in which all masses are taken to be the same, *i.e.* $\tilde{I}_{s,u} \mapsto I_{s,u}$. Using the tree-level amplitudes (4.3) it is not difficult to find all components of the C_s and C_u coefficients. For example,

$$\frac{1}{J_s}(C_s)_{ab}^{cd} = (A^{(0)2} + B^{(0)2} + 2C^{(0)}F^{(0)})\delta_a^c\delta_b^d + 2(A^{(0)}B^{(0)} - C^{(0)}F^{(0)})\delta_a^d\delta_b^c \quad (4.5)$$

$$\frac{1}{J_u}(C_u)_{ab}^{cd} = A^{(0)2}\delta_a^c\delta_b^d + 2(A^{(0)}B^{(0)} + B^{(0)2} - H^{(0)}K^{(0)})\delta_a^d\delta_b^c. \quad (4.6)$$

As mentioned in the previous section, the one-loop bubble integrals are such that the difference $C_s/J_s - C_u/J_u$ governs the logarithmic dependence on external momenta. While the complete expressions for C_s and C_u are not immediately transparent, their difference is simple –

$$\frac{C_s}{J_s} - \frac{C_u}{J_u} = + \frac{p^2 p'^2 (\varepsilon \varepsilon' - p p')}{(\varepsilon p' - \varepsilon' p)^2} \mathbb{1}, \quad (4.7)$$

i.e. it is proportional to the identity operator in field space; it is also independent of the gauge-choice parameter a , as expected. Using the values of the one-loop bubble integrals it follows that the one-loop worldsheet \mathbf{S} matrix in $\text{AdS}_5 \times \text{S}^5$ is

$$iT^{(1)} = i \left(\frac{1}{2} \left(-\frac{1}{\pi} \frac{p^2 p'^2 (\varepsilon \varepsilon' - p p')}{(\varepsilon p' - \varepsilon' p)^2} \ln \left| \frac{p'_-}{p_-} \right| \right) \mathbb{1} + \text{rational} \right) = i \left(\frac{1}{2} \hat{\theta}_{12}^{(1)} \mathbb{1} + \text{rational} \right). \quad (4.8)$$

We thus recover the general form of the one-loop \mathbf{S} matrix (2.13) with all the logarithmic terms¹⁶ given by the one-loop dressing phase in $\text{AdS}_5 \times \text{S}^5$, $\hat{\theta}_{12}^{(1)}$ in eq. (2.14). As mentioned in sec. 3, we can rewrite the one-loop \mathbf{S} matrix and separate the rational part while keeping an integral representation for the logarithmic dependence:

$$iT^{(1)} = \frac{1}{2} \frac{C_s}{J_s} (J_s I_s + 1) + \frac{1}{2} \frac{C_u}{J_u} J_u I_u + i\tilde{T}^{(1)}, \quad (4.9)$$

¹⁶We note here that, in line with the fact that the integrals we used are Lorentz invariant, the argument of the logarithm is Lorentz-invariant. The coefficient of the logarithm is not, however, and can also be written in terms of Lorentz invariants and the constant time-like vector n related to the choice of vacuum state: $\hat{\theta}_{12} = -1/\pi (\mathbf{p} \cdot \mathbf{p}' - n \cdot \mathbf{p} n \cdot \mathbf{p}')^2 (\mathbf{p} \cdot \mathbf{p}') / (\mathbf{p} \times \mathbf{p}')^2 \ln |p'_- / p_-|$. Such a rewriting is possible for all the other models we discuss in later sections.

where rational part¹⁷ of the one-loop S matrix, $\tilde{T}^{(1)}$, was defined in eq. (3.5).¹⁸ This decomposition will be useful in the next section where we discuss the construction of the logarithmic terms in the two-loop S matrix.

4.2 The logarithmic terms of the two-loop $\text{AdS}_5 \times \text{S}^5$ S matrix

The general form of the two-loop amplitudes in $\text{AdS}_5 \times \text{S}^5$ is given by (3.6) with all fields having the same mass; the coefficients are given by eqs. (3.7) and (3.12). We illustrate these formulae by writing out explicitly the components contributing to $A^{(2)}$ and $B^{(2)}$; they can also be rederived easily by iterating the general expressions for two-particle cuts in Appendix E:

$$\begin{aligned} \frac{1}{J_s^2}(C_a)_{ab}^{cd} &= (A_{s\text{-cut}}^{(1)}A^{(0)} + B_{s\text{-cut}}^{(1)}B^{(0)} + 2C_{s\text{-cut}}^{(1)}F^{(0)})\delta_a^c\delta_b^d \\ &\quad + (A_{s\text{-cut}}^{(1)}B^{(0)} + B_{s\text{-cut}}^{(1)}A^{(0)} - 2C_{s\text{-cut}}^{(1)}F^{(0)})\delta_a^d\delta_b^c, \\ &= (A^{(0)}A_{s\text{-cut}}^{(1)} + B^{(0)}B_{s\text{-cut}}^{(1)} + 2C^{(0)}F_{s\text{-cut}}^{(1)})\delta_a^c\delta_b^d \\ &\quad + (A^{(0)}B_{s\text{-cut}}^{(1)} + B^{(0)}A_{s\text{-cut}}^{(1)} - 2C^{(0)}F_{s\text{-cut}}^{(1)})\delta_a^d\delta_b^c, \end{aligned} \quad (4.10)$$

$$\begin{aligned} \frac{1}{J_u^2}(C_d)_{ab}^{cd} &= A_{u\text{-cut}}^{(1)}A^{(0)}\delta_a^c\delta_b^d \\ &\quad + (A_{u\text{-cut}}^{(1)}B^0 + B_{u\text{-cut}}^{(1)}A^{(0)} + 2B_{u\text{-cut}}^{(1)}B^{(0)} - 2H_{u\text{-cut}}^{(1)}K^{(0)})\delta_a^d\delta_b^c \\ &= A^{(0)}A_{u\text{-cut}}^{(1)}\delta_a^c\delta_b^d \\ &\quad + (A^{(0)}B_{u\text{-cut}}^{(1)} + B^{(0)}A_{u\text{-cut}}^{(1)} + 2B^{(0)}B_{u\text{-cut}}^{(1)} - 2H^{(0)}K_{u\text{-cut}}^{(1)})\delta_a^d\delta_b^c, \end{aligned} \quad (4.11)$$

$$\begin{aligned} \frac{1}{J_s J_u}(C_b)_{ab}^{cd} &= (A_{u\text{-cut}}^{(1)}A^{(0)} + B_{u\text{-cut}}^{(1)}B^{(0)} + 2C_{u\text{-cut}}^{(1)}F^{(0)})\delta_a^c\delta_b^d \\ &\quad + (A_{u\text{-cut}}^{(1)}B^{(0)} + B_{u\text{-cut}}^{(1)}A^{(0)} - 2C_{u\text{-cut}}^{(1)}F^{(0)})\delta_a^d\delta_b^c, \end{aligned} \quad (4.12)$$

$$\begin{aligned} \frac{1}{J_s J_u}(C_e)_{ab}^{cd} &= A_{s\text{-cut}}^{(1)}A^{(0)}\delta_a^c\delta_b^d \\ &\quad + (A_{s\text{-cut}}^{(1)}B^{(0)} + B_{s\text{-cut}}^{(1)}A^{(0)} + 2B_{s\text{-cut}}^{(1)}B^{(0)} - 2H_{s\text{-cut}}^{(1)}K^{(0)})\delta_a^d\delta_b^c, \end{aligned} \quad (4.13)$$

$$\begin{aligned} \frac{1}{J_s J_u}(C_c)_{ab}^{cd} &= (A^{(0)}A_{u\text{-cut}}^{(1)} + B^{(0)}B_{u\text{-cut}}^{(1)} + 2C^{(0)}F_{u\text{-cut}}^{(1)})\delta_a^c\delta_b^d \\ &\quad + (A^{(0)}B_{u\text{-cut}}^{(1)} + B^{(0)}A_{u\text{-cut}}^{(1)} - 2C^{(0)}F_{u\text{-cut}}^{(1)})\delta_a^d\delta_b^c, \end{aligned} \quad (4.14)$$

$$\begin{aligned} \frac{1}{J_s J_u}(C_f)_{ab}^{cd} &= A^{(0)}A_{s\text{-cut}}^{(1)}\delta_a^c\delta_b^d \\ &\quad + (A^{(0)}B_{s\text{-cut}}^{(1)} + B^{(0)}A_{s\text{-cut}}^{(1)} + 2B^{(0)}B_{s\text{-cut}}^{(1)} - 2H^{(0)}K_{s\text{-cut}}^{(1)})\delta_a^d\delta_b^c. \end{aligned} \quad (4.15)$$

The cuts of the one-loop amplitudes are given in terms of C_s and C_u coefficients, e.g.

$$iA_{s\text{-cut}}^{(1)}\delta_a^c\delta_b^d = \frac{1}{J_s}(C_s)_{ab}^{cd} \quad iA_{u\text{-cut}}^{(1)}\delta_a^c\delta_b^d = \frac{1}{J_u}(C_u)_{ab}^{cd}. \quad (4.16)$$

¹⁷We argued in footnote 12 that finiteness of rational terms in an integrable theory relies on the absence of tadpole integrals on external lines (sometimes known as snail graphs). One can check that the one-loop integral identified by the one-particle cut constructed from the four-point amplitude vanishes upon integration; this is in line with the expected absence of one-loop corrections to the dispersion relation of worldsheet fields in $\text{AdS}_5 \times \text{S}^5$. Thus, while we do not determine $\tilde{T}^{(1)}$ we nevertheless see that it is finite.

¹⁸We note that, since the difference of C_s and C_u is independent of the gauge-choice parameter a , only the last term, $\tilde{T}^{(1)}$, can depend on it.

Using the explicit expression for the tree-level S-matrix elements it is not difficult to check that

$$-2C_a + C_b + C_c - 2C_d + C_e + C_f = 0 , \quad (4.17)$$

as mentioned at the end of sec. 3. This implies the cancellation of all the double logarithms.¹⁹ This is consistent with the structure of the two-loop terms in eq. (2.13), since the two-loop correction to the BHL/BES dressing phase, $\hat{\theta}^{(2)}$, does not contain any double logarithms (see Appendix C).

The two remaining coefficients, $C_{s,\text{extra}}$ and $C_{u,\text{extra}}$, are given by eq. (3.12) and are determined by comparing the single two-particle cuts of the ansatz with the single two-particle cut of the two-loop amplitude. Their contribution to $A^{(2)}$ is

$$\begin{aligned} \frac{C_{s,\text{extra}}^A}{J_s} &= 2i(i)^2 (A^{(0)}(i\tilde{A}^{(1)}) + B^{(0)}(i\tilde{B}^{(1)}) + C^{(0)}(i\tilde{F}^{(1)}) + F^{(0)}(i\tilde{C}^{(1)})) \\ &\quad + \frac{C_a^A}{J_s^2} - \frac{1}{2} \left(\frac{C_b^A}{J_s} + \frac{C_c^A}{J_s} \right) I_t , \end{aligned} \quad (4.18)$$

$$\frac{C_{u,\text{extra}}^A}{J_u} = 2(i)^2 i A^{(0)}(i\tilde{A}^{(1)}) + \frac{1}{2} \left(\frac{C_e^A}{J_s J_u} + \frac{C_f^A}{J_s J_u} \right) - \frac{1}{2} \left(\frac{C_e^A}{J_s} + \frac{C_f^A}{J_s} \right) I_t . \quad (4.19)$$

C_x^Y denotes the contribution of C_x to the coefficient Y in eq. (4.1) and $\tilde{A}^{(1)}$, *etc.* are the entries of $\tilde{T}^{(1)}$ defined in eq. (3.5).

Taking their difference and reconstructing the combination on the second line of eq. (3.16) we observe the cancellation the diagonal term $\tilde{A}^{(1)}$ which potentially contains rational terms undetermined by symmetries. After some amount of algebra which makes use of the explicit form of the rational terms²⁰ extracted from the exact S matrix

$$\begin{aligned} \hat{B}^{(1)} &= i(A^{(0)} + D^{(0)})B^{(0)} + \frac{i}{8}aJB^{(0)} \\ \hat{C}^{(1)} &= \frac{i}{2}(A^{(0)} + D^{(0)})C^{(0)} + ib(p+p')C^{(0)} \\ \hat{F}^{(1)} &= \frac{i}{2}(A^{(0)} + D^{(0)})F^{(0)} - ib(p+p')F^{(0)} \end{aligned} \quad (4.20)$$

we find that the second line of eq. (3.16) vanishes identically. Repeating the same steps for the coefficients of all the other tensor structures one can confirm that

$$\frac{C_{s,\text{extra}}}{J_s} - \frac{C_{u,\text{extra}}}{J_u} + \frac{i}{8\pi J_s} (C_b + C_c - C_e - C_f) = 0 . \quad (4.21)$$

The combination of integral coefficients on first line of eq. (3.16) contributing to $iA^{(2)}$ can be written as

$$-4\pi i \mathcal{C}_{\ln^1}^A = \frac{1}{J^2} (2C_a^A - C_b^A - C_c^A) = -2(iA^{(0)}) \left(\frac{C_s^A}{J_s} - \frac{C_u^A}{J_u} \right) ; \quad (4.22)$$

¹⁹Such cancellations between terms captured by different cuts is characteristic to un-ordered scattering amplitudes and was previously observed in *e.g.* higher-dimensional supergravity theories.

²⁰We notice here that the off-diagonal rational terms of the one-loop amplitudes are proportional to the tree-level S-matrix elements.

Similar expressions hold for all the other entries. Combining them and using the value of the parenthesis above from eq. (4.7) we find that the $\text{AdS}_5 \times \text{S}^5$ two-loop S matrix is

$$iT^{(2)} = -\frac{1}{2} \left(-\frac{1}{\pi} \frac{p^2 p'^2 (\varepsilon \varepsilon' - p p')}{(\varepsilon' p - \varepsilon p')^2} \ln \frac{p'_-}{p_-} \right) T^{(0)} + \text{rational} = -\frac{1}{2} \hat{\theta}_{12}^{(1)} T^{(0)} + \text{rational} . \quad (4.23)$$

It reproduces the logarithmic terms in eq. (2.13) and thus gives strong support to the exponentiation of the one-loop logarithms as in eq. 2.12.

We have therefore demonstrated that the generalized unitarity method carried out in two dimensions, together with consequences of symmetries of the theory, can be used to efficiently determine all terms with logarithmic dependence on external momenta through at least two loop level. We shall now discuss other interesting worldsheet theories related to gauge/string dualities and in some cases find novel results regarding their scattering matrices.

5 On the S matrix for strings in $\text{AdS}_4 \times \text{CP}^3$

A worldsheet theory that bears close similarity to the $\text{AdS}_5 \times \text{S}^5$ string is that of type IIA string theory on $\text{AdS}_4 \times \text{CP}^3$. This is dual to the ABJM theory [8] and there is evidence for its integrability. Quantized around a BMN-like null geodesics, its worldsheet spectrum consists of eight bosons and eight fermions four of each being light ($m^2 = 1/4$) and four being heavy ($m^2 = 1$). It was argued from a worldsheet perspective [48] that the heavy excitations are unstable and decay into a pair of light excitations. The bosonic worldsheet S matrix was found in [35]. In a spin-chain picture the heavy excitations are interpreted as composite and do not exist as asymptotic states; thus, the relevant S matrix scatters only the eight light excitations, organized in two representations of $PSU(2|2) \times U(1)$, typically called the A and the B particles/multiplets. For our discussion we shall use the small momentum limit of this S-matrix. As discussed in sec. 2, the different masses of excitations and the properties of the S matrix guarantee that this truncation is perturbatively consistent through at least two loops.

In [12] a proposal was made for the Bethe equations to all-loop orders. The dressing phase for $\text{AdS}_4 \times \text{CP}^3$ was suggested to be similar to the one found for $\text{AdS}_5 \times \text{S}^5$ with the difference that the full S-matrix contained the squared dressing phase (with the two factors coming from the two factors in (2.1)); in the former case there is one dressing phase factor for each multiplet.

Based on these Bethe ansatz equations an S matrix was conjectured in [49]. The proposed S matrix has four sectors corresponding to the four ways one can pick the multiplets of the incoming and outgoing particles and it is reflectionless (*i.e.* particles of different multiplets will not exchange momentum):

$$S^{BB}(p, p') = S^{AA}(p, p') = S_0(p, p') \hat{S}(p, p'), \quad (5.1)$$

$$S^{AB}(p, p') = S^{BA}(p, p') = \tilde{S}_0(p, p') \hat{S}(p, p') . \quad (5.2)$$

The factors $S_0(p, p')$ and $\tilde{S}_0(p, p')$ are two (potentially different) dressing phases and $\hat{S}(p, p')$ is the $SU(2|2)$ invariant S matrix found in [47].

The reflectionless property of the S matrix implies that different sectors do not mix in the s - and u -channel cuts so it is sufficient to consider the cuts of a single $SU(2|2)$ -invariant S

matrix. Our computations in the previous section show that, at least through two loops, the dressing phase of such an S matrix would indeed have half of the logarithmic terms of the full BES/BHL phase, consistent with the expectation of [12] thus providing a non-trivial check of that proposal.

An alternative S matrix was considered in [50] and rejected as the resulting Bethe equations did not match two-loop gauge theory perturbative calculations [51]. It is also based on the $SU(2|2)$ invariant S matrix, with the only important difference that it is no longer reflectionless. We can also see from the perspective of generalized unitarity that such an S matrix is not consistent with worldsheet perturbation theory truncated to the fields with $m^2 = 1/4$.

To this end let us consider the scattering of two A -type scalars into two A -type fermions. Since this is an off-diagonal matrix element and since (assuming integrability and $SU(2|2)$ symmetry) quantum corrections can yield logarithmic terms only in the dressing phase, eq. (2.13) implies that at one-loop level this matrix element should have no logarithms. For a reflectionless S matrix this is realized by the s - and u -channel cuts being equal. Allowing reflections in the S matrix changes the number of particles crossing the u -channel cut (from only A -type particles to both A - and B -type particles) while not affecting the s -channel cut. Thus, a reflection-containing S matrix is not consistent with worldsheet perturbation theory truncated to the light fields.

6 The S matrix for strings in $AdS_3 \times S^3 \times S^3 \times S^1$

The Green-Schwarz string in $AdS_3 \times S^3 \times S^3 \times S^1$ [52] was shown in [4] to be related to the \mathbf{Z}_4 -graded supercoset $D(2, 1; \alpha) \otimes D(2, 1; \alpha) / (SO(1, 2) \times SO(3) \times SO(3))$ and an extra free boson, where α is related to the radii of the various factors by

$$\alpha = \frac{R_{AdS}^2}{R_{S_1^3}^2} = 1 - \frac{R_{AdS}^2}{R_{S_2^3}^2} . \quad (6.1)$$

The BMN limit was studied in [53] and it was found that the perturbative worldsheet perturbative consists of

$$\begin{array}{ll} \text{two bosons and two fermions of } m = 1 & \text{two bosons and two fermions of } m = 0 \\ \text{two bosons and two fermions of } m = \alpha & \text{two bosons and two fermions of } m = 1 - \alpha . \end{array}$$

One massless mode corresponds to excitations on S^1 and the other to an excitation shared between the two three-spheres. Certain entries of the tree-level S matrix for particles with α -dependent masses were found in [24].

The symmetry group of the light-cone gauge-fixed worldsheet theory is $PSU(1|1)^2$. As discussed in [15], the states with α -dependent masses may be written as a tensor product of left excitations (L), $|\phi\rangle$ and $|\psi\rangle$, and the conjugate right excitations (R), $|\bar{\phi}\rangle$ and $|\bar{\psi}\rangle$; they transform (in conjugate representations) under one or the other factors of the symmetry group.²¹ The two representations are no longer decoupled in the centrally-extended $PSU(1|1)^2$ and, while $PSU(1|1)^2$ -invariant, the S matrix describes nontrivial LL , RR , LR and RL scattering.

²¹The L and R excitations have been denoted by $+$ and $-$ in related a context [34]. We mention here again that these excitations are not directly related to left- and right- worldsheet motion.

This sector of massive states was discussed from a spin-chain perspective in [15, 17] and independently in [18] where the symmetry-determined parts of the S-matrices were proposed. The direct tree-level calculation of [24] favors the S matrix proposed in [15, 17]. The one-loop correction to the dressing phase was found indirectly in [22], by comparing one-loop corrections to energies of semiclassical states to the energy predictions of the Asymptotic Bethe Ansatz based on the exact S matrix.

As described in [17], the proposed S-matrices scatter only states with normalized masses $m = \alpha$ and $m = (1 - \alpha)$. One of their characteristic feature is that individual masses are conserved in a scattering process. Let us consider some generic scattering process:

$$|\mathcal{X}_{p_{in}}^{(in)} \mathcal{X}_{p'_{in}}^{(in)}\rangle \rightarrow |\mathcal{X}_{p_{out}}^{(out)} \mathcal{X}_{p'_{out}}^{(out)}\rangle, \quad (6.2)$$

where we also assume that $\frac{p_{in}}{m_{in}} > \frac{p'_{in}}{m'_{in}}, \frac{p_{out}}{m_{out}} > \frac{p'_{out}}{m'_{out}}$ ²² where m denote the normalized masses of particles. When all the masses are the same momentum and energy conservation leads to the solution $p_{in} = p_{out}, p'_{in} = p'_{out}$ plus another solution that does not satisfy the assumed ordering of momenta.

When masses are not the same one can still have a similar solution to the conservation equations, $p_{in} = p_{out}, p'_{in} = p'_{out}, m_{in} = m_{out} \neq m'_{in} = m'_{out}$ but there is also another solution where the outgoing momenta are not equal to the incoming momenta. The proposed S-matrices are reflectionless, *i.e.* they forbid this second possibility.

The scattering of states in different multiplets of the $PSU(1|1)^2$ symmetry group is also reflectionless, *i.e.* in the scattering a left and a right excitation momentum is not exchanged between particles even if they have the same mass.

6.1 The tree-level S matrix

Entries of the tree-level worldsheet S matrix were constructed in [24] for general α . The result reproduces the small momentum limit of the full S-matrices of [15]; an alternative S matrix was proposed in [18]; both the S matrix of Borsato, Ohlsson Sax and Sfondrini (BOSS) [15] and that of Ahn and Bombardelli (AB) [18] are included in Appendix H. In a notation close to that used in $AdS_5 \times S^5$, we parametrize the T as follows:

$$\begin{aligned} T_{\phi\phi}^{\phi\phi} &= A_{LL}, & T_{\psi\psi}^{\psi\psi} &= D_{LL}, & T_{\phi\psi}^{\phi\psi} &= G_{LL}, \\ T_{\phi\psi}^{\psi\phi} &= H_{LL}, & T_{\psi\phi}^{\phi\psi} &= K_{LL}, & T_{\psi\phi}^{\psi\phi} &= L_{LL}, \\ T_{\phi\bar{\phi}}^{\phi\bar{\phi}} &= A_{LR}, & T_{\phi\bar{\phi}}^{\psi\bar{\psi}} &= C_{LR}, & T_{\psi\bar{\psi}}^{\psi\bar{\psi}} &= D_{LR}, \\ T_{\phi\bar{\psi}}^{\phi\bar{\psi}} &= G_{LR}, & T_{\psi\bar{\psi}}^{\phi\bar{\phi}} &= F_{LR}, & T_{\phi\bar{\psi}}^{\psi\bar{\phi}} &= H_{LR}, \\ T_{\psi\bar{\phi}}^{\phi\bar{\psi}} &= K_{LR}, & T_{\psi\bar{\phi}}^{\psi\bar{\phi}} &= L_{LR}. \end{aligned} \quad (6.3)$$

The lower indices denote the incoming state and the upper indices denote the outgoing state. Here we did not assign an index for the mass of the excitation; such an index can be added without difficulty. Apart from fields, the indices L and R should carry the same index related to

²²This condition is equivalent to assuming a definite sign for the Jacobian $J = \varepsilon'p - p'\varepsilon$.

the mass. Each component has an expansion in the appropriate worldsheet coupling constant \hat{g}^{-1} similar to eq. 4.2, *e.g.*

$$A_{LL} = \frac{1}{\hat{g}} A_{LL}^{(0)} + \frac{1}{\hat{g}^2} A_{LL}^{(1)} + \dots \quad (6.4)$$

The tree-level worldsheet S-matrix elements, $A_{LL}^{(0)}$ *etc.*, follow from [24] or can be extracted from the small momentum expansion of the exact S matrix [15] with additional minus signs for the scattering of two fermions. The latter approach also yields the symmetry-determined rational terms at higher loops.

We will not need the explicit form of the diagonal tree-level matrix elements, so we will not list them here. Their only property that is important for our calculation is that they are related by

$$A^{(0)} + D^{(0)} - G^{(0)} - L^{(0)} = 0 \quad , \quad (6.5)$$

for any choice of masses and for any choice of L and R states.

With these preparations, the tree-level entries of the BOSS S-matrix that we will need are [24, 15]:

$$H_{LL}^{(0)\text{BOSS}} = K_{LL}^{(0)\text{BOSS}} = \frac{1}{2} \frac{pp'}{\varepsilon'p - p'\varepsilon} \frac{(\varepsilon + m)(\varepsilon' + m') - pp'}{\sqrt{(\varepsilon + m)(\varepsilon' + m')}} \quad , \quad (6.6)$$

$$C_{LR}^{(0)\text{BOSS}} = F_{LR}^{(0)\text{BOSS}} = \frac{1}{2} \frac{\sqrt{(\varepsilon + m)(\varepsilon' + m')}(\varepsilon'p - \varepsilon p' + p'm - pm')}{\varepsilon'p - p'\varepsilon} \quad , \quad (6.7)$$

$$H_{LR}^{(0)\text{BOSS}} = K_{LR}^{(0)\text{BOSS}} = 0 \quad . \quad (6.8)$$

Here we combined all different choices of masses in a single expression; different mass sectors correspond to different choices of $m, m' = \alpha, 1 - \alpha$; also, ε and ε' are the standard relativistic energies for particles of masses m and m' , respectively.

Similarly, expanding the non-diagonal entries of the AB S matrix and combining different mass sectors we find:

$$H_{LL}^{(0)\text{AB}} = H_{LR}^{(0)\text{AB}} = \frac{1}{2} \frac{pp'}{\varepsilon'p - p'\varepsilon} \frac{(\varepsilon + m)(\varepsilon' + m') - pp'}{(\varepsilon + m)} \quad , \quad (6.9)$$

$$C_{LR}^{(0)\text{AB}} = F_{LR}^{(0)\text{AB}} = 0 \quad , \quad (6.10)$$

$$K_{LL}^{(0)\text{AB}} = K_{LR}^{(0)\text{AB}} = \frac{1}{2} \frac{pp'}{\varepsilon'p - p'\varepsilon} \frac{(\varepsilon + m)(\varepsilon' + m') - pp'}{(\varepsilon + m')} \quad . \quad (6.11)$$

Notice that although the functions H_{LL} and K_{LL} are different in eqs. (6.8) and (6.11), their product remains the same. This observation will be useful shortly and leads to the equality of the one-loop corrections in the LL/RR sectors of the two S-matrices.

6.2 The logarithmic terms of the one-loop $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$ S-matrix

Let us now illustrate the calculation of the one-loop S matrix following the general discussion in sec. 3. Each multi-index A, B, C, \dots there stands for the triplet (field label, mass, sector), *e.g.*

(ϕ, m, L) ; the grade of the index $[A]$ is the grade of the field label (0/1 for bosons/fermions). The general expression for one-loop four-point amplitudes is (3.1) with integrals having different masses. They obey the relations

$$J_s \tilde{I}_s - J_u \tilde{I}_u = -\frac{2i}{\pi} \left(\ln \left| \frac{p'_-}{p_-} \right| - \ln \left| \frac{m'}{m} \right| \right) - 1, \quad (6.12)$$

$$J_s \tilde{I}_s + J_u \tilde{I}_u + 1 = 0, \quad (6.13)$$

which implies that the logarithmic dependence of the S matrix is governed by the difference $C_s/J_s - C_u/J_u$. It is instructive to consider separately the LL and LR sectors and find the one-loop correction in the various possible sectors. In the process we shall also expand the slightly cryptic multi-index form of the one-loop coefficients (3.2).

6.2.1 The LL and RR sectors

The tree-level amplitudes in the BOSS and AB S-matrices have the same vanishing entries; therefore the one-loop coefficients (3.2) have formally similar expressions:

$$\begin{aligned} \frac{1}{J_s}(C_{s,LL})_{\phi\phi}^{\phi\phi} &= A_{LL}^{(0)2} & \frac{1}{J_u}(C_{u,LL})_{\phi\phi}^{\phi\phi} &= A_{LL}^{(0)2} - H_{LL}^{(0)} K_{LL}^{(0)} \\ \frac{1}{J_s}(C_{s,LL})_{\psi\psi}^{\psi\psi} &= D_{LL}^{(0)2} & \frac{1}{J_u}(C_{u,LL})_{\psi\psi}^{\psi\psi} &= D_{LL}^{(0)2} - H_{LL}^{(0)} K_{LL}^{(0)} \\ \frac{1}{J_s}(C_{s,LL})_{\phi\psi}^{\phi\psi} &= G_{LL}^{(0)2} + H_{LL}^{(0)} K_{LL}^{(0)} & \frac{1}{J_u}(C_{u,LL})_{\phi\psi}^{\phi\psi} &= G_{LL}^{(0)2} \\ \frac{1}{J_s}(C_{s,LL})_{\psi\phi}^{\phi\psi} &= G_{LL}^{(0)} H_{LL}^{(0)} + H_{LL}^{(0)} L_{LL}^{(0)} & \frac{1}{J_u}(C_{u,LL})_{\psi\phi}^{\phi\psi} &= D_{LL}^{(0)} H_{LL}^{(0)} + H_{LL}^{(0)} A_{LL}^{(0)} \\ \frac{1}{J_s}(C_{s,LL})_{\phi\psi}^{\psi\phi} &= L_{LL}^{(0)} K_{LL}^{(0)} + K_{LL}^{(0)} G_{LL}^{(0)} & \frac{1}{J_u}(C_{u,LL})_{\phi\psi}^{\psi\phi} &= A_{LL}^{(0)} K_{LL}^{(0)} + K_{LL}^{(0)} D_{LL}^{(0)} \\ \frac{1}{J_s}(C_{s,LL})_{\psi\phi}^{\psi\phi} &= L_{LL}^{(0)2} + H_{LL}^{(0)} K_{LL}^{(0)} & \frac{1}{J_u}(C_{u,LL})_{\psi\phi}^{\psi\phi} &= L_{LL}^{(0)2} \end{aligned} \quad (6.14)$$

It is easy to see that the difference $(C_s/J_s - C_u/J_u)$, governing the logarithmic dependence on external momenta, depends only on the product $H_{LL}^{(0)} K_{LL}^{(0)}$ which is the same for both S-matrices (6.8) and (6.11). We therefore find that

$$\frac{C_{s,LL}^{\text{BOSS}}}{J_s} - \frac{C_{u,LL}^{\text{BOSS}}}{J_u} = \frac{C_{s,LL}^{\text{AB}}}{J_s} - \frac{C_{u,LL}^{\text{AB}}}{J_u} = \frac{p^2(p')^2(\mathbf{p} \cdot \mathbf{p}' + mm')}{2(\varepsilon'p - p'\varepsilon)^2} \mathbb{1}, \quad (6.15)$$

where $\mathbb{1}$ is the identity matrix in field space. This in turn implies that the one-loop correction to the LL sector of both S-matrices is

$$iT_{LL}^{(1)} = i \left(\frac{1}{2} \left(-\frac{1}{\pi} \frac{p^2(p')^2(\mathbf{p} \cdot \mathbf{p}' + mm')}{2(\varepsilon'p - p'\varepsilon)^2} \left(\ln \left| \frac{p'_-}{p_-} \right| - \ln \left| \frac{m'}{m} \right| \right) \right) \mathbb{1} + \text{rational} \right). \quad (6.16)$$

This expression reproduces the small momentum limit of the LL dressing phase of [22] for general values of m and m' , see eq. (H.22). The additional factor of 1/2 is reminiscent of $\text{AdS}_4 \times \text{CP}^3$ as the dressing phase to which we are comparing comes from a factorized S matrix.

The calculation in the RR sector is completely equivalent for both S matrices. While consistent with the BOSS S matrix, the presence of logarithms appears to contradict the conjecture [18] that in the sector with two different masses, $m \neq m'$, there is no dressing phase. This implies that the S matrix proposed there is not the S matrix of a quantum field theory.

6.2.2 The LR and RL sectors

Due to the structure of the tree-level AB S matrix, in particular the vanishing of $C_{LR}^{(0)}$ and $F_{LR}^{(0)}$ as well as the fact that $H_{LR}^{(0)} = H_{LL}^{(0)}$ and $K_{LR}^{(0)} = K_{LL}^{(0)}$, the calculation of the one-loop S matrix in the LR/RL sectors is identical to the one in the LL/RR sectors in the previous section and $iT_{LR}^{(1),AB}$ is given by the right-hand side of eq. (6.16).

Using the vanishing entries of the *BOSS* tree-level S matrix, the various components of the coefficients C_s and C_u describing its one-loop corrections are given by (3.2):

$$\begin{aligned}
\frac{1}{J_s}(C_{s,LR}^{\text{BOSS}})_{\phi\bar{\phi}}^{\phi\bar{\phi}} &= A_{LR}^{(0)2} + C_{LR}^{(0)}F_{LR}^{(0)} & \frac{1}{J_u}(C_{u,LR}^{\text{BOSS}})_{\phi\bar{\phi}}^{\phi\bar{\phi}} &= A_{LR}^{(0)2} \\
\frac{1}{J_s}(C_{s,LR}^{\text{BOSS}})_{\phi\bar{\phi}}^{\psi\bar{\psi}} &= A_{LR}^{(0)}C_{LR}^{(0)} + C_{LR}^{(0)}D_{LR}^{(0)} & \frac{1}{J_u}(C_{u,LR}^{\text{BOSS}})_{\phi\bar{\phi}}^{\psi\bar{\psi}} &= G_{LR}^{(0)}C_{LR}^{(0)} + C_{LR}^{(0)}L_{LR}^{(0)} \\
\frac{1}{J_s}(C_{s,LR}^{\text{BOSS}})_{\psi\bar{\psi}}^{\psi\bar{\psi}} &= D_{LR}^{(0)2} + C_{LR}^{(0)}F_{LR}^{(0)} & \frac{1}{J_u}(C_{u,LR}^{\text{BOSS}})_{\psi\bar{\psi}}^{\psi\bar{\psi}} &= D_{LR}^{(0)2} \\
\frac{1}{J_s}(C_{s,LR}^{\text{BOSS}})_{\psi\bar{\psi}}^{\phi\bar{\phi}} &= D_{LR}^{(0)}F_{LR}^{(0)} + F_{LR}^{(0)}A_{LR}^{(0)} & \frac{1}{J_u}(C_{u,LR}^{\text{BOSS}})_{\psi\bar{\psi}}^{\phi\bar{\phi}} &= L_{LR}^{(0)}F_{LR}^{(0)} + F_{LR}^{(0)}G_{LR}^{(0)} \\
\frac{1}{J_s}(C_{s,LR}^{\text{BOSS}})_{\phi\bar{\psi}}^{\phi\bar{\psi}} &= G_{LR}^{(0)2} & \frac{1}{J_u}(C_{u,LR}^{\text{BOSS}})_{\phi\bar{\psi}}^{\phi\bar{\psi}} &= G_{LR}^{(0)2} - C_{LR}^{(0)}F_{LR}^{(0)} \\
\frac{1}{J_s}(C_{s,LR}^{\text{BOSS}})_{\psi\bar{\phi}}^{\psi\bar{\phi}} &= L_{LR}^{(0)2} & \frac{1}{J_u}(C_{u,LR}^{\text{BOSS}})_{\psi\bar{\phi}}^{\psi\bar{\phi}} &= L_{LR}^{(0)2} - C_{LR}^{(0)}F_{LR}^{(0)}
\end{aligned} \tag{6.17}$$

From here the difference $C_s/J_s - C_u/J_u$ that governs the S matrix' logarithmic dependence on external momenta is

$$\frac{C_s^{\text{BOSS}}}{J_s} - \frac{C_u^{\text{BOSS}}}{J_u} = \frac{p^2(p')^2(\mathbf{p} \cdot \mathbf{p}' - mm')}{2(\varepsilon'p - p'\varepsilon)^2} \mathbb{1} . \tag{6.18}$$

where $\mathbb{1}$ is the identity matrix in field space. In turn this implies that the one-loop correction to the *BOSS* S matrix is

$$iT_{LR}^{(1),\text{BOSS}} = i \left(\frac{1}{2} \left(-\frac{1}{\pi} \frac{p^2(p')^2(\mathbf{p} \cdot \mathbf{p}' - mm')}{2(\varepsilon'p - p'\varepsilon)^2} \left(\ln \left| \frac{p'_-}{p_-} \right| - \ln \left| \frac{m'}{m} \right| \right) \right) \mathbb{1} + \text{rational} \right) . \tag{6.19}$$

This expression reproduces the small momentum limit of the LR dressing phase of [22] for general values of m and m' , see eq. (H.23). The additional factor of 1/2 has the same origin as in the LL and RR sectors.

7 The S matrix for strings in $\text{AdS}_3 \times \mathbf{S}^3 \times \mathbf{T}^4$

$\text{AdS}_3 \times \mathbf{S}^3 \times \mathbf{T}^4$ can be sourced by a mixture of RR and NSNS fluxes [54, 55, 56]. The Green-Schwarz string in such a background was constructed in [52]; the $\text{AdS}_3 \times \mathbf{S}^3$ part is described by the supercoset $PSU(1,1|2) \times PSU(1,1|2)/(SU(1,1) \times SU(2))$ and it is classically integrable [57]. This model was further studied in [34] where the tree-level four-point S matrix was found for a generic mixture of both types of fluxes.

In this section we will compute the logarithmic part of the one-loop S matrix in the presence of both NSNS and RR fluxes. For a pure RR flux background we extract the rational terms from the all-loop symmetry-determined S matrix found in [16] and compute the two-loop logarithmic terms.

The massive worldsheet spectrum in $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$ consists of eight modes (four bosons and four fermions) which are organized in two bifundamental representations of the light-cone gauge symmetry group $SU(1|1)^2 \times SU(1|1)^2$; they are denoted by [34]

$$\begin{aligned} |y_+\rangle &= |\phi\rangle \otimes |\phi\rangle & |z_+\rangle &= |\psi\rangle \otimes |\psi\rangle & |\zeta_+\rangle &= |\phi\rangle \otimes |\psi\rangle & |\chi_+\rangle &= |\psi\rangle \otimes |\phi\rangle, \\ |y_-\rangle &= |\bar{\phi}\rangle \otimes |\bar{\phi}\rangle & |z_-\rangle &= |\bar{\psi}\rangle \otimes |\bar{\psi}\rangle & |\zeta_-\rangle &= |\bar{\phi}\rangle \otimes |\bar{\psi}\rangle & |\chi_-\rangle &= |\bar{\psi}\rangle \otimes |\bar{\phi}\rangle. \end{aligned} \quad (7.1)$$

We slightly adjusted the notation, in particular $\phi_+ \rightarrow \phi$, $\phi_- \rightarrow \bar{\phi}$, to make it match the pure RR flux S matrix of [16] where (ϕ, ψ) are the excitations in the L sector and $(\bar{\phi}, \bar{\psi})$ are the excitations in the R sector. The symmetry group implies that the S matrix has the usual factorized structure (2.1)

$$\mathbf{S} = \mathbf{S}_{su(1|1)^2} \otimes \mathbf{S}_{su(1|1)^2}. \quad (7.2)$$

As in $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$, each of the two \mathbf{S} -matrix factors has four sectors: LL , LR , RL and RR (or $(++)$, $(+-)$, $(-+)$, $(--)$ in the notation of [34]).

7.1 String theory tree-level S-matrices

The T matrix is parameterized in the same way as in $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$, *i.e.*

$$\begin{aligned} T_{\phi\phi}^{\phi\phi} &= A_{LL}, & T_{\psi\psi}^{\psi\psi} &= D_{LL}, & T_{\phi\psi}^{\phi\psi} &= G_{LL}, \\ T_{\phi\psi}^{\psi\phi} &= H_{LL}, & T_{\psi\phi}^{\phi\psi} &= K_{LL}, & T_{\psi\phi}^{\psi\phi} &= L_{LL}, \\ T_{\phi\bar{\phi}}^{\phi\bar{\phi}} &= A_{LR}, & T_{\phi\bar{\phi}}^{\psi\bar{\psi}} &= C_{LR}, & T_{\psi\bar{\psi}}^{\psi\bar{\psi}} &= D_{LR}, \\ T_{\phi\bar{\psi}}^{\phi\bar{\psi}} &= G_{LR}, & T_{\psi\bar{\psi}}^{\phi\bar{\phi}} &= F_{LR}, & T_{\psi\bar{\phi}}^{\psi\bar{\phi}} &= L_{LR}, \end{aligned} \quad (7.3)$$

with each coefficient having an expansion in the inverse string tension. For the background supported by a mixture of RR and NSNS flux the LL sector of the tree-level S matrix is [34]

$$T_{LL}^{(0)}|\phi\phi'\rangle = \frac{1}{2}(l_1 + c)|\phi\phi'\rangle, \quad T_{LL}^{(0)}|\phi\psi'\rangle = \frac{1}{2}(l_3 + c)|\phi\psi'\rangle - l_5|\psi\phi'\rangle, \quad (7.4)$$

$$T_{LL}^{(0)}|\psi\psi'\rangle = \frac{1}{2}(-l_1 + c)|\psi\psi'\rangle, \quad T_{LL}^{(0)}|\psi\phi'\rangle = \frac{1}{2}(-l_3 + c)|\psi\phi'\rangle - l_5|\phi\psi'\rangle, \quad (7.5)$$

and the LR sector is

$$T_{LR}^{(0)}|\phi\bar{\psi}'\rangle = \frac{1}{2}(l_3 + c)|\phi\bar{\psi}'\rangle, \quad T_{LR}^{(0)}|\phi\bar{\phi}'\rangle = \frac{1}{2}(l_2 + c)|\phi\bar{\phi}'\rangle + l_4|\psi\bar{\psi}'\rangle, \quad (7.6)$$

$$T_{LR}^{(0)}|\psi\bar{\phi}'\rangle = \frac{1}{2}(-l_3 + c)|\psi\bar{\phi}'\rangle, \quad T_{LR}^{(0)}|\psi\bar{\psi}'\rangle = \frac{1}{2}(-l_2 + c)|\psi\bar{\psi}'\rangle + l_4|\phi\bar{\phi}'\rangle. \quad (7.7)$$

In the following we shall need only the expressions for l_4 and l_5 :

$$l_4 = -\frac{pp'}{2(p+p')} \left(\sqrt{(\varepsilon_+ + p)(\varepsilon'_- + p')} - \sqrt{(\varepsilon_+ - p)(\varepsilon'_- - p')} \right), \quad (7.8)$$

$$l_5 = -\frac{pp'}{2(p-p')} \left(\sqrt{(\varepsilon_+ + p)(\varepsilon'_+ + p')} + \sqrt{(\varepsilon_+ - p)(\varepsilon'_+ - p')} \right), \quad (7.9)$$

$$\varepsilon_{\pm} = \sqrt{(p \pm q)^2 + 1 - q^2}. \quad (7.10)$$

Here q is a measure of the ratio between the NSNS and RR fluxes; $q \rightarrow 0$ yields the pure RR flux theory. The other two sectors of the S matrix can be found by exchanging ε_+ and ε_- as well as barred and un-barred fields. We note that for $q \rightarrow 0$ the form of the S matrix in the LL and LR sectors is the same as the $\alpha \rightarrow 1$ limit of the LL and LR sectors of the BOSS S matrix [15, 24]. As before, the tree-level S-matrix coefficients are related by:

$$A^{(0)} + D^{(0)} - G^{(0)} - L^{(0)} = 0, \quad (7.11)$$

in all sectors. This identity will be useful for the consistency of the construction of loop amplitudes we now discuss.

7.2 One-loop logarithmic terms for mixed RR/NSNS $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$

The one-loop amplitudes take again the general form (3.1); however, because the dispersion relations in the presence of a non vanishing q are not the standard relativistic ones, the integrals \tilde{I} have a slightly different interpretation: the space-like component of the momentum of a propagator is shifted by $\pm q$ for a field in the L/R sector. Moreover, the masses are $(1 - q^2)$. These integrals may be interpreted as regular integrals with equal masses evaluated at shifted external momenta²³:

$$\begin{aligned} \tilde{I}(p, p')_{LL} &= I(p + q, p' + q) \\ \tilde{I}(p, p')_{LR} &= I(p + q, p' - q). \end{aligned} \quad (7.12)$$

Their expressions follow immediately from Appendix F. Due to the modified dispersion relations, the Jacobian factors (2.23) are modified from their usual form $J = 4(p\varepsilon' - p'\varepsilon)$ to J_{LL} and J_{LR} following from (2.22) for the dispersion relation in eq. (7.10):

$$J_{LL} = 4((p + q)\varepsilon'_+ - (p' + q)\varepsilon_+) \quad J_{LR} = 4((p + q)\varepsilon'_- - (p' - q)\varepsilon_+). \quad (7.13)$$

Since all masses are equal, the multi-indices used in sec. 3 now represent the pair (field label, sector).

The non-zero integral coefficients in the LL sector are given by (3.2)

$$\begin{aligned} \frac{1}{J_{s,LL}}(C_{s,LL})_{\phi\phi} &= A_{LL}^{(0)2} & \frac{1}{J_{u,LL}}(C_{u,LL})_{\phi\phi} &= A_{LL}^{(0)2} - H_{LL}^{(0)}K_{LL}^{(0)} \\ \frac{1}{J_{s,LL}}(C_{s,LL})_{\psi\psi} &= D_{LL}^{(0)2} & \frac{1}{J_{u,LL}}(C_{u,LL})_{\psi\psi} &= D_{LL}^{(0)2} - H_{LL}^{(0)}K_{LL}^{(0)} \\ \frac{1}{J_{s,LL}}(C_{s,LL})_{\phi\psi} &= G_{LL}^{(0)2} + H_{LL}^{(0)}K_{LL}^{(0)} & \frac{1}{J_{u,LL}}(C_{u,LL})_{\phi\psi} &= G_{LL}^{(0)2} \\ \frac{1}{J_{s,LL}}(C_{s,LL})_{\psi\phi} &= G_{LL}^{(0)}H_{LL}^{(0)} + H_{LL}^{(0)}L_{LL}^{(0)} & \frac{1}{J_{u,LL}}(C_{u,LL})_{\psi\phi} &= D_{LL}^{(0)}H_{LL}^{(0)} + H_{LL}^{(0)}A_{LL}^{(0)} \\ \frac{1}{J_{s,LL}}(C_{s,LL})_{\phi\phi} &= L_{LL}^{(0)}K_{LL}^{(0)} + K_{LL}^{(0)}G_{LL}^{(0)} & \frac{1}{J_{u,LL}}(C_{u,LL})_{\phi\phi} &= A_{LL}^{(0)}K_{LL}^{(0)} + K_{LL}^{(0)}D_{LL}^{(0)} \\ \frac{1}{J_{s,LL}}(C_{s,LL})_{\psi\phi} &= L_{LL}^{(0)2} + H_{LL}^{(0)}K_{LL}^{(0)} & \frac{1}{J_{u,LL}}(C_{u,LL})_{\psi\phi} &= L_{LL}^{(0)2} \end{aligned} \quad (7.14)$$

From here follows that the difference of C_s and C_u coefficients relevant for the logarithmic terms in (3.3) in the LL sector is

$$\frac{C_{s,LL}}{J_s} - \frac{C_{u,LL}}{J_u} = \frac{p^2(p')^2}{2(p - p')^2} \left(\varepsilon_+\varepsilon'_+ + pp' + \sqrt{(1 + 2pq)(1 + 2p'q)} \right) \mathbb{1} \quad (7.15)$$

²³To derive these expressions it is necessary to parametrize the integrals to respect the $p \leftrightarrow p'$ symmetry of the graph.

which depends only on the product $H^{(0)}K^{(0)}$.

The nonzero integral coefficients in the LR sector are given by

$$\begin{aligned}
\frac{1}{J_{s,LR}}(C_{s,LR})_{\phi\bar{\phi}}^{\phi\bar{\phi}} &= A_{LR}^{(0)2} + C_{LR}^{(0)}F_{LR}^{(0)} & \frac{1}{J_{u,LR}}(C_{u,LR})_{\phi\bar{\phi}}^{\phi\bar{\phi}} &= A_{LR}^{(0)2} \\
\frac{1}{J_{s,LR}}(C_{s,LR})_{\phi\bar{\phi}}^{\psi\bar{\psi}} &= A_{LR}^{(0)}C_{LR}^{(0)} + C_{LR}^{(0)}D_{LR}^{(0)} & \frac{1}{J_{u,LR}}(C_{u,LR})_{\phi\bar{\phi}}^{\psi\bar{\psi}} &= G_{LR}^{(0)}C_{LR}^{(0)} + C_{LR}^{(0)}L_{LR}^{(0)} \\
\frac{1}{J_{s,LR}}(C_{s,LR})_{\psi\bar{\psi}}^{\phi\bar{\phi}} &= D_{LR}^{(0)2} + C_{LR}^{(0)}F_{LR}^{(0)} & \frac{1}{J_{u,LR}}(C_{u,LR})_{\psi\bar{\psi}}^{\phi\bar{\phi}} &= D_{LR}^{(0)2} \\
\frac{1}{J_{s,LR}}(C_{s,LR})_{\psi\bar{\psi}}^{\psi\bar{\psi}} &= D_{LR}^{(0)}F_{LR}^{(0)} + F_{LR}^{(0)}A_{LR}^{(0)} & \frac{1}{J_{u,LR}}(C_{u,LR})_{\psi\bar{\psi}}^{\psi\bar{\psi}} &= L_{LR}^{(0)}F_{LR}^{(0)} + F_{LR}^{(0)}G_{LR}^{(0)} \\
\frac{1}{J_{s,LR}}(C_{s,LR})_{\phi\bar{\psi}}^{\phi\bar{\psi}} &= G_{LR}^{(0)2} & \frac{1}{J_{u,LR}}(C_{u,LR})_{\phi\bar{\psi}}^{\phi\bar{\psi}} &= G_{LR}^{(0)2} - C_{LR}^{(0)}F_{LR}^{(0)} \\
\frac{1}{J_{s,LR}}(C_{s,LR})_{\psi\bar{\phi}}^{\psi\bar{\phi}} &= L_{LR}^{(0)2} & \frac{1}{J_{u,LR}}(C_{u,LR})_{\psi\bar{\phi}}^{\psi\bar{\phi}} &= L_{LR}^{(0)2} - C_{LR}^{(0)}F_{LR}^{(0)} .
\end{aligned} \tag{7.16}$$

The corresponding difference (3.3) relevant for the logarithmic terms in this sector is then

$$\frac{C_{s,LR}}{J_{s,LR}} - \frac{C_{u,LR}}{J_{u,LR}} = \frac{p^2(p')^2}{2(p+p')^2} \left(\varepsilon_+\varepsilon'_- + pp' - \sqrt{(1+2pq)(1-2p'q)} \right) \mathbb{1} , \tag{7.17}$$

and depends only on $C^{(0)}F^{(0)}$.

Using the integrals (7.12) to reconstruct the one-loop S matrix we find:

$$iT_{LL}^{(1)} = i \left(\frac{1}{2} \left(- \frac{1}{\pi} \frac{p^2(p')^2 \left(\varepsilon_+\varepsilon'_+ + pp' + \sqrt{(1+2pq)(1+2p'q)} \right)}{2(p-p')^2} \ln \left| \frac{\varepsilon'_+ - p' - q}{\varepsilon_+ - p - q} \right| \right) \mathbb{1} + \text{rat.} \right) , \tag{7.18}$$

$$iT_{LR}^{(1)} = i \left(\frac{1}{2} \left(- \frac{1}{\pi} \frac{p^2(p')^2 \left(\varepsilon_+\varepsilon'_- + pp' - \sqrt{(1+2pq)(1-2p'q)} \right)}{2(p+p')^2} \ln \left| \frac{\varepsilon'_- - p' + q}{\varepsilon_+ - p - q} \right| \right) \mathbb{1} + \text{rat.} \right) , \tag{7.19}$$

i.e. only the diagonal entries are corrected. This result is in line with the expectation (2.13) that at one loop all logarithmic corrections are proportional to the identity matrix. Comparison with that equation identifies the coefficients of $(i/2 \mathbb{1})$ with the one-loop dressing phases $\theta_{LL}^{(1)}$ and $\theta_{LR}^{(1)}$ in the LL and LR sectors.

In the limit $q \rightarrow 0$ the momentum dependence of eqs. (7.18) and (7.19) becomes the same as that of the small momentum limit of the phase factors found in [22], eqs. (H.22) and (H.23).²⁴ The near-flat space limit of these phases was also found through a Feynman graph calculation in [24]. An additional factor of 1/2 relates to the fact that the phase factor of [22] receives contributions from both factors of the factorized S matrix (2.1). This pattern is the same as in $\text{AdS}_5 \times S^5$.

7.3 Two-loop logarithmic terms in $\text{AdS}_3 \times S^3 \times T^4$ with RR flux

The two-loop logarithmic terms for mixed NRNR and RR flux background can be found as soon as the symmetry-determined part of the S matrix becomes available [58]. For the $q = 0$

²⁴The simplest way to see this is to solve the on shell conditions and write both expressions in terms of p_- and p'_- .

case the relevant S matrix was recently suggested in [16]; up to the choice of mass scale it is the same as the S matrix in $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$ which is included in Appendix H.

The two-loop amplitude has the general form (3.6) with coefficients given by (3.7) and (3.12). As for $q \neq 0$, the multi-indices are pairs (field label, sector). We discuss separately the LL and LR sectors, focusing on the S-matrix element $A^{(2)}$. The contribution of the C coefficients to it will be denoted by $C^{A_{LL}}$ and $C^{A_{LR}}$.

7.3.1 The LL and RR sectors

Expressed in terms of cuts of the one-loop amplitudes, the six coefficients $C_a^{A_{LL}}$ (3.7) are

$$\begin{aligned} \frac{1}{J^2} C_a^{A_{LL}} &= A_{LL}^{(1)} \text{ s-cut } A_{LL}^{(0)}, \\ \frac{1}{J^2} C_d^{A_{LL}} &= A_{LL}^{(1)} \text{ u-cut } A_{LL}^{(0)} - \frac{1}{2} H_{LL}^{(1)} \text{ u-cut } K_{LL}^{(0)} - \frac{1}{2} H_{LL}^{(0)} K_{LL}^{(1)} \text{ u-cut}, \\ \frac{1}{J^2} C_b^{A_{LL}} &= A_{LL}^{(1)} \text{ u-cut } A_{LL}^{(0)}, \\ \frac{1}{J^2} C_e^{A_{LL}} &= A_{LL}^{(1)} \text{ s-cut } A_{LL}^{(0)} - H_{LL}^{(1)} \text{ s-cut } K_{LL}^{(0)}, \\ \frac{1}{J^2} C_c^{A_{LL}} &= A_{LL}^{(1)} \text{ u-cut } A_{LL}^{(0)}, \\ \frac{1}{J^2} C_f^{A_{LL}} &= A_{LL}^{(1)} \text{ s-cut } A_{LL}^{(0)} - H_{LL}^{(0)} K_{LL}^{(1)} \text{ s-cut}. \end{aligned} \quad (7.20)$$

We recall that the cuts of the one-loop coefficients are given by the relevant components of the C_s and C_u coefficients (7.14). It is trivial to check that

$$C_{\ln^2}^{A_{LL}} \propto -2C_a^{A_{LL}} + C_b^{A_{LL}} + C_c^{A_{LL}} - 2C_d^{A_{LL}} + C_e^{A_{LL}} + C_f^{A_{LL}} = 0, \quad (7.21)$$

which implies that the double logarithms cancel out of the $A_{LL}^{(2)}$. The same holds for all other components of the two-loop S matrix.

We note here that eqs. (7.20) hold for $q \neq 0$ as well; the resulting two-loop integral coefficients $C_a \dots C_f$ also obey the relation (7.21), guaranteeing the absence of double logarithms in the dressing phases of the S matrix in the mixed NSNS/RR-flux background as well.

The single-log terms in A_{LL} have the same form in terms of the C coefficients as given in eq. (3.16). The remaining coefficients $C_{s,\text{extra}}$ and $C_{u,\text{extra}}$, obtained by matching the single two-particle cuts of the ansatz onto the single two-particle cuts of the four-point amplitudes, are given by (3.12)

$$\begin{aligned} \frac{C_{s,\text{extra}}^{A_{LL}}}{J} &= -i A_{LL}^{(0)} \frac{C_s^{A_{LL}}}{J} - \frac{1}{2} \left(\frac{C_b^{A_{LL}}}{J^2} + \frac{C_c^{A_{LL}}}{J^2} \right) J_u I_t + 2 A_{LL}^{(0)} \tilde{A}_{LL}^{(1)}, \\ \frac{C_{u,\text{extra}}^{A_{LL}}}{J} &= -i A_{LL}^{(0)} \frac{C_s^{A_{LL}}}{J} + \frac{1}{2} i H_{LL}^{(0)} \frac{C_s^{K_{LL}}}{J} + \frac{1}{2} i K_{LL}^{(0)} \frac{C_s^{H_{LL}}}{J} - \frac{1}{2} \left(\frac{C_e^{A_{LL}}}{J^2} + \frac{C_f^{A_{LL}}}{J^2} \right) J_u I_t \\ &\quad + 2 A_{LL}^{(0)} \tilde{A}_{LL}^{(1)} - H_{LL}^{(0)} \tilde{K}_{LL}^{(1)} - K_{LL}^{(0)} \tilde{H}_{LL}^{(1)}, \end{aligned} \quad (7.22)$$

with the coefficients with tilde being the entries of the rational part, $\tilde{T}^{(1)}$, of the one-loop S matrix introduced in eq. (3.5). The difference of the C_{extra} coefficients that enters eq. (3.16) is

then

$$\begin{aligned} \frac{C_{s,\text{extra}}^{A_{LL}}}{J} - \frac{C_{u,\text{extra}}^{A_{LL}}}{J} = & \frac{1}{2} \left(\frac{C_e^{A_{LL}}}{J^2} + \frac{C_f^{A_{LL}}}{J^2} - \frac{C_b^{A_{LL}}}{J^2} - \frac{C_c^{A_{LL}}}{J^2} \right) J_u I_t \\ & - \frac{1}{2} H_{LL}^{(0)} (2\tilde{K}_{LL}^{(1)} - (i)^2 K_{LL}^{(1)}{}_{s\text{-cut}}) \\ & - \frac{1}{2} K_{LL}^{(0)} (2\tilde{H}_{LL}^{(1)} - (i)^2 H_{LL}^{(1)}{}_{s\text{-cut}}) , \end{aligned} \quad (7.23)$$

where we used $J_s = J_u$. We notice the cancellation of the diagonal rational term $\tilde{A}^{(1)}$ which carries the only dependence on the undetermined rational function Φ . The relevant one-loop rational terms are²⁵

$$\tilde{H}_{LL}^{(1)} = \hat{H}_{LL}^{(1)} = \frac{i}{2} \left(A_{LL}^{(0)} + D_{LL}^{(0)} \right) H_{LL}^{(0)} + \frac{i}{4} (1 + 4b)(p - p') H_{LL}^{(0)}, \quad (7.24)$$

$$\tilde{K}_{LL}^{(1)} = \hat{K}_{LL}^{(1)} = \frac{i}{2} \left(A_{LL}^{(0)} + D_{LL}^{(0)} \right) K_{LL}^{(0)} - \frac{i}{4} (1 + 4b)(p - p') K_{LL}^{(0)}. \quad (7.25)$$

Together with the one-loop coefficients $H^{(1)}$ and $K^{(1)}$ they imply that the terms on the second line of (3.16) vanish identically and that the only contribution to the single-log terms in $A_{LL}^{(2)}$ comes from the first line of that equation:

$$-4\pi i C_{\ln^1}^{A_{LL}} = \frac{1}{J^2} (2C_a^{A_{LL}} - C_b^{A_{LL}} - C_c^{A_{LL}}) = -2(iA_{LL}^{(0)}) \left(\frac{C_s^{A_{LL}}}{J_s} - \frac{C_u^{A_{LL}}}{J_u} \right); \quad (7.26)$$

the expression inside the parentheses can be read off of eq. (7.15) in the limit $q \rightarrow 0$.

Repeating the calculation for other entries of the two-loop S-matrix²⁶ we find similar results except that $(iA_{LL}^{(0)})$ is replaced with the tree-level value of that entry and the parenthesis is replaced with some combination of the entries of $C_s/J_s - C_u/J_u$ whose value equals that of $C_s^{A_{LL}}/J_s - C_u^{A_{LL}}/J_u$. Thus, the two-loop S matrix in the LL sector is

$$iT_{LL}^{(2)} = -\frac{1}{2} T^{(0)} \theta_{LL}^{(1)} + \text{rational} \quad (7.27)$$

The RR sector S matrix is completely identical.

7.3.2 The LR and RL sectors

The construction of the two-loop S matrix in the LR sector is very similar except for the specifics related to the tree-level S matrix in this sector. The six coefficients determined by

²⁵We notice here that, similarly to the $\text{AdS}_5 \times S^5$ S matrix, the one-loop off-diagonal rational terms are proportional to the tree-level S-matrix elements.

²⁶Both here and in the LR sector, the identity

$$\hat{A}^{(1)} + \hat{D}^{(1)} - \hat{G}^{(1)} - \hat{L}^{(1)} + \frac{1}{2} A_{s\text{-cut}}^{(1)} + \frac{1}{2} D_{s\text{-cut}}^{(1)} - \frac{1}{2} G_{s\text{-cut}}^{(1)} - \frac{1}{2} L_{s\text{-cut}}^{(1)} = 0 .$$

is necessary for finding the off-diagonal entries of the S matrix.

maximal cuts are (3.7)

$$\begin{aligned}
\frac{1}{J^2} C_a^{A_{LR}} &= A_{LR}^{(1)} s\text{-cut} A_{LR}^{(0)} + \frac{1}{2} C_{LR}^{(0)} F_{LR}^{(1)} s\text{-cut} + \frac{1}{2} C_{LR}^{(1)} s\text{-cut} F_{LR}^{(0)} , \\
\frac{1}{J^2} C_d^{A_{LR}} &= A_{LR}^{(1)} s\text{-cut} A_{LR}^{(0)} , \\
\frac{1}{J^2} C_b^{A_{LR}} &= A_{LR}^{(1)} u\text{-cut} A_{LR}^{(0)} + C_{LR}^{(1)} u\text{-cut} F_{LR}^{(0)} , \\
\frac{1}{J^2} C_e^{A_{LR}} &= A_{LR}^{(1)} s\text{-cut} A_{LR}^{(0)} , \\
\frac{1}{J^2} C_c^{A_{LR}} &= A_{LR}^{(1)} u\text{-cut} A_{LR}^{(0)} + C_{LR}^{(0)} F_{LR}^{(1)} u\text{-cut} , \\
\frac{1}{J^2} C_f^{A_{LR}} &= A_{LR}^{(1)} s\text{-cut} A_{LR}^{(0)} .
\end{aligned} \tag{7.28}$$

As before, the cuts of the one-loop S-matrix elements are given in terms of the appropriate components of C_s and C_u . It is easy to check that

$$C_{\ln^2}^{A_{LR}} = -2C_a^{A_{LR}} + C_b^{A_{LR}} + C_c^{A_{LR}} - 2C_d^{A_{LR}} + C_e^{A_{LR}} + C_f^{A_{LR}} = 0 \tag{7.29}$$

and thus the double logarithms cancel out. The same holds for all other components of the S matrix.

As in the LL sector, the integral coefficients $C_a \dots C_f$ can also be found in a mixed RR/NSNS background and they obey the condition (7.29). Thus, double logarithms are also absent from the two-loop S matrix in this more general case.

The remaining two coefficients, $C_{s,\text{extra}}^{A_{LR}}$ and $C_{u,\text{extra}}^{A_{LR}}$, follow from (3.12):

$$\frac{C_{s,\text{extra}}^{A_{LR}}}{J} = -i A_{LR}^{(0)} \frac{C_s^{A_{LR}}}{J} - \frac{1}{2} i C^{(0)} \frac{C_s^{F_{LR}}}{J} - \frac{1}{2} i F^{(0)} \frac{C_s^{C_{LR}}}{J} - \frac{1}{2} \left(\frac{C_b^{A_{LR}}}{J} + \frac{C_c^{A_{LR}}}{J} \right) I_t \tag{7.30}$$

$$\begin{aligned}
&+ 2A_{LR}^{(0)} \tilde{A}_{LR}^{(1)} + C_{LR}^{(0)} \tilde{F}_{LR}^{(1)} + \tilde{C}_{LR}^{(1)} F_{LR}^{(0)} \\
\frac{C_{u,\text{extra}}^{A_{LR}}}{J} &= -i A_{LR}^{(0)} \frac{C_s^{A_{LR}}}{J} - \frac{1}{2} \left(\frac{C_e^{A_{LR}}}{J} + \frac{C_f^{A_{LR}}}{J} \right) I_t + 2A_{LR}^{(0)} \tilde{A}_{LR}^{(1)} .
\end{aligned} \tag{7.31}$$

In their difference, which enters eq. (3.16), we notice again the cancellation of the diagonal one-loop rational term $\tilde{A}_{LR}^{(1)}$, which guarantees that all the single-logarithms are independent of the rational part Φ of the dressing phase, see eq (3.5).

With the rational terms extracted from the symmetry-determined one-loop S matrix²⁷

$$\tilde{C}_{LR}^{(1)} = \hat{C}_{LR}^{(1)} = \frac{i}{2} \left(A_{LR}^{(0)} + D_{LR}^{(0)} \right) C_{LR}^{(0)} + \frac{i}{4} (1 + 4b)(p + p') C_{LR}^{(0)} , \tag{7.32}$$

$$\tilde{F}_{LR}^{(1)} = \hat{F}_{LR}^{(1)} = \frac{i}{2} \left(A_{LR}^{(0)} + D_{LR}^{(0)} \right) F_{LR}^{(0)} - \frac{i}{4} (1 + 4b)(p + p') F_{LR}^{(0)} , \tag{7.33}$$

we find that the second line on the right-hand side of eq. (3.16) vanishes identically and that

$$-4\pi i C_{\ln^1}^{A_{LR}} = \frac{1}{2J^2} (2C_a^{A_{LR}} - C_b^{A_{LR}} - C_c^{A_{LR}}) = -2(iA_{LR}^{(0)}) \left(\frac{C_s^{A_{LR}}}{J_s} - \frac{C_u^{A_{LR}}}{J_u} \right) . \tag{7.34}$$

The value of the parenthesis can be read off eq. (7.17) in the limit $q \rightarrow 0$.

²⁷As in the LL sector and in $\text{AdS}_5 \times S^5$, the rational terms are proportional to the tree-level S-matrix elements.

Repeating the calculation for other entries of the two-loop S matrix we find similar results upon using all relations between the two-loop integral coefficients. Thus, the two-loop S matrix in the LR sector is

$$iT_{LR}^{(2)} = -\frac{1}{2} T^{(0)} \theta_{LR}^{(1)} + \text{rational} \quad (7.35)$$

This result, as well as eq. (7.27), supports the exponentiation of the one-loop dressing phase in all sectors of the theory and thus provide support for the quantum integrability of the theory through two loops. The RL sector S matrix is completely identical.

8 Concluding remarks

In this paper we discussed the calculation of the logarithmic terms in the S-matrices of two-dimensional integrable quantum field theories using the generalized unitarity method and its refinements. The calculation of unitarity cuts was carried out in two dimensions and thus it potentially drops terms with completely rational dependence on external momenta. By supplying the rational lower-loop terms determined by symmetries we can recover all the higher-loop logarithms. We illustrated this approach with one- and two-loop calculations in worldsheet theories relevant to gauge/string dualities – string theory in $\text{AdS}_5 \times S^5$, $\text{AdS}_4 \times \text{CP}^3$, $\text{AdS}_3 \times S^3 \times S^3 \times S^1$ and $\text{AdS}_3 \times S^3 \times T^4$.

Using this approach we successfully recovered the known logarithmic terms in the $\text{AdS}_5 \times S^5$ S matrix and thus provided evidence that, in this approach, the structure of the S matrix is that implied by integrability. We also computed the logarithmic terms of the one-loop S matrix for strings in $\text{AdS}_3 \times S^3 \times S^3 \times S^1$ for a general value of α and reproduced the dressing phase obtained by matching the one-loop energy calculation of semiclassical states with the predictions of the Asymptotic Bethe Ansatz. We also discussed string theory in $\text{AdS}_3 \times S^3 \times T^4$ sourced by a mixture of RR and NSNS fluxes and found the logarithmic terms in the dressing phase. In the limit of vanishing NSNS flux and in the near-flat space limit we recovered the result of [24]. For vanishing NSNS flux we have also computed the logarithmic terms in the two-loop S matrix and showed that they all come from the exponentiation of the dressing phases. For the mixed case one can see that the double logarithms cancel out; it should not be difficult to test this structure once a symmetry-determined S matrix becomes available.

The calculations described in this paper can be extended to higher loops. While for strings in $\text{AdS}_5 \times S^5$ all necessary information is available, potential subtleties arise for other backgrounds. We argued in sec. 2 that through two loops, unitarity-based quantum calculations in subsectors of theories with reflectionless S-matrices capture all logarithmic terms. A more detailed analysis is necessary to ascertain whether this continues to hold at higher loops. An important part will likely be played by the structure of higher-loop integrals. Absence of additional logarithms in the dressing phase at two loops suggests that the pole structure of the S matrix is the expected one through this order and that, to this order, the spectrum of bound states is the known one. Higher-loop calculations would provide further tests in this direction.

In our calculations we relied on symmetries to determine the off-diagonal rational parts of one-loop S-matrix elements. One may continue to do so at higher loops as well. It would

of course be desirable to have an independent derivation of both diagonal and off-diagonal rational terms, the former being unconstrained by symmetries. We expect that they can be found though use of dimensional regularization, albeit in that case one would need to supply local counterterms to restore *e.g.* integrability and perhaps also other symmetries.

A direct determination of all rational terms would presumably clarify the interplay between regularization and symmetries in the Green-Schwarz string and it may provide a means to discuss from an S matrix perspective the existence of integrability anomalies such as those in the bosonic CP^{n-1} model [59]. It would also be an important ingredient in the unitarity-based construction of other interesting worldsheet quantities, such as worldsheet form factors [60] or correlation functions of operators, perhaps along the lines of [61]. Similar methods can be used to construct the two-point off-shell Green's function of worldsheet fields and thus find the loop corrections to the dispersion relation as well as the corrected propagator residue needed for the determination of complete higher-loop scattering matrices.

Note added: While the writing of this paper was being completed we received [62] by L. Bianchi, V. Forini, B. Hoare where the idea of using generalized unitarity for the calculation of worldsheet S-matrices was proposed independently and applied to one-loop calculations in several integrable relativistic theories and in string theory in $AdS_5 \times S^5$. A natural prescription for the coefficient of the t -channel bubble integral was shown to lead to complete one-loop S-matrices. In contrast, we determine rational terms from symmetry considerations. We also discussed the application of generalized unitarity in worldsheet theories truncated to a subset of fields, obtained general expressions for amplitude through two loops and discussed in detail string theory in $AdS_5 \times S^5$, $AdS_4 \times CP^3$, $AdS_3 \times S^3 \times S^3 \times S^1$ and $AdS_3 \times S^3 \times T^4$.

Acknowledgments

We would like to thank Z. Bern, J.J. Carrasco, L. Dixon, H. Johansson, A. Tseytlin and C. Vergu for useful discussions and L. Bianchi, V. Forini, B. Hoare and A. Tseytlin for comments on the draft. We especially thank D. Kosower for collaboration at initial stages of this project and on related subjects and for detailed comments on the draft. We also thank V. Forini, B. Hoare and L. Bianchi for pre-publication copy of [62]. This work is supported by the US Department of Energy under contract DE-SC0008745.

A On the definition of the tree-level S-matrix elements

The worldsheet tree-level S-matrix elements are computed in a slightly different normalization than in usual four-dimensional calculations. In this appendix we summarize the relevant definitions, which hopefully will make transparent the translation to four-dimensional conventions.

- Fields are expanded in creation and annihilation operators as, (ϕ is a generic real scalar field)

$$\phi = \int \frac{dk_1}{(2\pi)\sqrt{2k_0}} \left(a_{k_1} e^{ik \cdot x} + a_{k_1}^\dagger e^{-ik \cdot x} \right) \quad (\text{A.1})$$

i.e. with the measure missing a factor of $(\sqrt{2k_0})^{-1}$ compared to the standard relativistic normalization.

- The commutation relations are missing a factor of $2k_0$ compared to the standard relativistic normalization

$$[a_k, a_{k'}^\dagger] = 2\pi\delta(k - k') \quad (\text{A.2})$$

This is of course a consequence of the previous item.

- States are defined in the usual way, e.g.

$$a_{k_{1,1}}^\dagger a_{k_{1,2}}^\dagger |0\rangle \quad (\text{A.3})$$

- Momentum conservation $\delta^{(2)}(\sum_k p_k)$ was solved as [42] in the presence of the on-shell condition for external states

$$\begin{aligned} & \delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4) \delta(p_1 + p_2 - p_3 - p_4) \\ &= \frac{\varepsilon_1 \varepsilon_2}{\varepsilon_2 p_1 - \varepsilon_1 p_2} (\delta(p_1 - p_3) \delta(p_2 - p_4) + \delta(p_1 - p_4) \delta(p_2 - p_3)) \end{aligned} \quad (\text{A.4})$$

This expression assumes that $p_1 > p_2$; otherwise the Jacobian factor is negative.

B Beisert's $\text{SU}(2|2)$ spin-chain S matrix

Beisert's spin-chain S matrix [46] is defined by its action on two-particle states:

$$\mathbf{S}^B |\phi_a \phi'_b\rangle = \mathbf{A}^B |\phi'_{\{a} \phi_b\}\rangle + \mathbf{B}^B |\phi'_{[a} \phi_{b]}\rangle + \frac{1}{2} \mathbf{C}^B \epsilon_{ab} \epsilon^{\alpha\beta} |\mathcal{Z}^- \psi'_\alpha \psi_\beta\rangle, \quad (\text{B.1})$$

$$\mathbf{S}^B |\psi_\alpha \psi'_\beta\rangle = \mathbf{D}^B |\psi'_{\{\alpha} \psi_{\beta]\rangle} + \mathbf{E}^B |\psi'_{[\alpha} \psi_{\beta]\rangle} + \frac{1}{2} \mathbf{F}^B \epsilon_{\alpha\beta} \epsilon^{ab} |\mathcal{Z}^+ \phi'_a \phi_b\rangle, \quad (\text{B.2})$$

$$\mathbf{S}^B |\phi_a \psi'_\beta\rangle = \mathbf{G}^B |\psi'_\beta \phi_a\rangle + \mathbf{H}^B |\phi'_a \psi_\beta\rangle, \quad (\text{B.3})$$

$$\mathbf{S}^B |\psi_\alpha \phi'_b\rangle = \mathbf{K}^B |\psi'_\alpha \phi_b\rangle + \mathbf{L}^B |\phi'_b \psi_\alpha\rangle. \quad (\text{B.4})$$

Its coefficients are fixed by the requirement that it exhibits $PSU(2|2)$ symmetry; they are

$$\begin{aligned}
\mathbf{A}^B &= S_{pp'}^0 \frac{x_{p'}^+ - x_p^-}{x_{p'}^- - x_p^+}, \\
\mathbf{B}^B &= S_{pp'}^0 \frac{x_{p'}^+ - x_p^-}{x_{p'}^- - x_p^+} \left(1 - 2 \frac{1 - \frac{1}{x_p^+ x_{p'}^-}}{1 - \frac{1}{x_p^+ x_{p'}^+}} \frac{x_{p'}^- - x_p^-}{x_{p'}^+ - x_p^-} \right), \\
\mathbf{C}^B &= S_{pp'}^0 \frac{2\gamma_p \gamma_{p'}}{x_p^+ x_{p'}^+} \frac{1}{1 - \frac{1}{x_p^+ x_{p'}^+}} \frac{x_{p'}^- - x_p^-}{x_{p'}^+ - x_p^+}, \\
\mathbf{D}^B &= -S_{pp'}^0, \\
\mathbf{E}^B &= -S_{pp'}^0 \left(1 - 2 \frac{1 - \frac{1}{x_p^- x_{p'}^+}}{1 - \frac{1}{x_p^- x_{p'}^-}} \frac{x_{p'}^+ - x_p^+}{x_{p'}^- - x_p^+} \right), \\
\mathbf{F}^B &= -S_{pp'}^0 \frac{2}{\gamma_p \gamma_{p'} x_p^- x_{p'}^-} \frac{(x_p^+ - x_p^-)(x_{p'}^+ - x_{p'}^-)}{1 - \frac{1}{x_p^- x_{p'}^-}} \frac{x_{p'}^+ - x_p^+}{x_{p'}^- - x_p^+}, \\
\mathbf{G}^B &= S_{pp'}^0 \frac{x_{p'}^+ - x_p^+}{x_{p'}^- - x_p^+}, & \mathbf{H}^B &= S_{pp'}^0 \frac{\gamma_p}{\gamma_{p'}} \frac{x_{p'}^+ - x_p^-}{x_{p'}^- - x_p^+}, \\
\mathbf{K}^B &= S_{pp'}^0 \frac{\gamma_{p'}}{\gamma_p} \frac{x_p^+ - x_p^-}{x_{p'}^- - x_p^+}, & \mathbf{L}^B &= S_{pp'}^0 \frac{x_{p'}^- - x_p^-}{x_{p'}^- - x_p^+},
\end{aligned} \tag{B.5}$$

where

$$\gamma_p = |x_p^- - x_p^+|^{1/2} \tag{B.6}$$

and

$$x_p^\pm = \frac{\pi e^{\pm \frac{i}{2}p}}{\sqrt{\lambda} \sin \frac{p}{2}} \left(1 + \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p}{2}} \right). \tag{B.7}$$

are the Zhukowsky variables.

The overall phase factor S^0 is undetermined by the algebraic construction. The one that correctly reproduces the semiclassical string spectrum via Bethe equations is

$$S_{pp'}^0 = \frac{1 - \frac{1}{x_{p'}^+ x_p^-}}{1 - \frac{1}{x_{p'}^- x_p^+}} e^{i\theta(p,p')} \tag{B.8}$$

with the dressing phase θ given to the leading order in $1/\sqrt{\lambda}$ by [39]

$$\begin{aligned}
\theta(p, p') &= \frac{\sqrt{\lambda}}{2\pi} \sum_{m, m'=\pm} m m' \hat{\chi}(x_p^m, x_{p'}^{m'}), \\
\hat{\chi}(x, y) &= (x - y) \left[\frac{1}{xy} + \left(1 - \frac{1}{xy} \right) \ln \left(1 - \frac{1}{xy} \right) \right].
\end{aligned} \tag{B.9}$$

C Strong coupling expansion of the $\text{AdS}_5 \times \text{S}^5$ dressing phase

The general form of the dressing phase in terms of higher local conserved charges is [63]:

$$\theta_{12} = \sum_{r=2}^{\infty} \sum_{s=r+1}^{\infty} c_{r,s}(\sqrt{\lambda})(q_r(x_1^{\pm})q_s(x_2^{\pm}) - q_r(x_2^{\pm})q_s(x_1^{\pm})) \quad (\text{C.1})$$

$$q_r(x^{\pm}) = \frac{i}{r-1} \left(\frac{1}{(x^+)^{r-1}} - \frac{1}{(x^-)^{r-1}} \right) \quad (\text{C.2})$$

This may be rewritten as [41]

$$\begin{aligned} \theta_{12} &= \chi(x_1^+, x_2^+) - \chi(x_1^+, x_2^-) - \chi(x_1^-, x_2^+) + \chi(x_1^-, x_2^-) \\ &\quad - \chi(x_2^+, x_1^+) + \chi(x_2^+, x_1^-) + \chi(x_2^-, x_1^+) - \chi(x_2^-, x_1^-) \end{aligned} \quad (\text{C.3})$$

$$\chi(x_1, x_2) = \sum_{r=2}^{\infty} \sum_{s=r+1}^{\infty} \frac{-c_{r,s}}{(r-1)(s-1)} \frac{1}{x_1^{r-1} x_2^{s-1}}. \quad (\text{C.4})$$

The coefficients $c_{r,s}$ depend on the coupling constant \hat{g} as

$$c_{rs} = \sum_{n=0}^{\infty} \hat{g}^{1-n} c_{rs}^{(n)} \rightarrow \chi(x_1, x_2) = \sum_{n=0}^{\infty} \hat{g}^{1-n} \chi^{(n)}(x_1, x_2) \rightarrow \theta_{12} = \sum_{n=0}^{\infty} \hat{g}^{1-n} \theta_{12}^{(n)} \quad (\text{C.5})$$

$$\begin{aligned} \theta_{12}^{(n)} &= \chi^{(n)}(x_1^+, x_2^+) - \chi^{(n)}(x_1^+, x_2^-) - \chi^{(n)}(x_1^-, x_2^+) + \chi^{(n)}(x_1^-, x_2^-) \\ &\quad - \chi^{(n)}(x_2^+, x_1^+) + \chi^{(n)}(x_2^+, x_1^-) + \chi^{(n)}(x_2^-, x_1^+) - \chi^{(n)}(x_2^-, x_1^-). \end{aligned} \quad (\text{C.6})$$

For string theory in $\text{AdS}_5 \times \text{S}^5$ the coefficients $c_{rs}^{(0)}$ were found in [6] and $c_{rs}^{(1)}$ in [64]. An all-loop proposal was put forward in [25].

The functions $\chi^{(n)}$ to various loop orders are:

$$\chi^{(0)}(x_1, x_2) = -\frac{1}{x_2} + \left(\frac{1}{x_2} - x_1 \right) \ln \left(1 - \frac{1}{x_1 x_2} \right) \quad (\text{C.7})$$

$$\begin{aligned} \chi^{(1)}(x_1, x_2) &= -\frac{1}{2\pi} \text{Li}_2 \frac{\sqrt{x_1} - 1/\sqrt{x_2}}{\sqrt{x_1} - \sqrt{x_2}} - \frac{1}{2\pi} \text{Li}_2 \frac{\sqrt{x_1} + 1/\sqrt{x_2}}{\sqrt{x_1} + \sqrt{x_2}} \\ &\quad + \frac{1}{2\pi} \text{Li}_2 \frac{\sqrt{x_1} + 1/\sqrt{x_2}}{\sqrt{x_1} - \sqrt{x_2}} + \frac{1}{2\pi} \text{Li}_2 \frac{\sqrt{x_1} - 1/\sqrt{x_2}}{\sqrt{x_1} + \sqrt{x_2}} \end{aligned} \quad (\text{C.8})$$

$$\chi^{(2)}(x_1, x_2) = -\frac{x_2}{24(x_1 x_2 - 1)(x_2^2 - 1)} \quad (\text{C.9})$$

$$\chi^{(3)}(x_1, x_2) = 0 \quad (\text{C.10})$$

$$\chi^{(4)}(x_1, x_2) = -\frac{x_2^3 + 4x_2^5 - 9x_1 x_2^6 + x_2^7 + 3x_1^2 x_2^7 - 3x_1 x_2^8 + 3x_1^2 x_2^9}{720(x_1 x_2 - 1)^3(x_2^2 - 1)^5} \quad (\text{C.11})$$

$$\chi^{(2k+1)}(x_1, x_2) = 0 \quad (\text{C.12})$$

An all-order integral representation of the dressing phase was put forward in [65].

We emphasize here that the only logarithmic terms in the dressing phase have a one-loop origin. This will nevertheless lead to logarithmic terms at higher loops in worldsheet perturbation theory. The relation between the spin chain strong coupling expansion and worldsheet perturbation theory was mentioned in sec. 2 will be reviewed in Appendix D.

D The $\text{AdS}_5 \times \text{S}^5$ worldsheet \mathbf{S} matrix from the spin-chain \mathbf{S} matrix

The symmetry of the world sheet theory is $PSU(2|2)^2$; together with the expected integrability they imply that the world sheet S -matrix factorizes as

$$\mathbb{S} = \mathbf{S} \otimes \mathbf{S} \quad (\text{D.1})$$

where each factor transforms under a copy of $PSU(2|2)$; neither factor is the \mathbf{S} matrix of any obvious excitations of the worldsheet theory. The four-point tree-level worldsheet \mathbf{S} matrix in $\text{AdS}_5 \times \text{S}^5$ was computed in [26] and this factorization was verified.

Since only $SU(2) \times SU(2) \subset PSU(2|2)$ is a manifest symmetry of the gauge-fixed worldsheet theory, \mathbf{S} may be parametrized in terms of ten unknown functions of the momenta p and p' of the two incoming particles:²⁸.

$$\begin{aligned} \mathbf{S}_{ab}^{cd} &= \mathbf{A} \delta_a^c \delta_b^d + \mathbf{B} \delta_a^d \delta_b^c & , & & \mathbf{S}_{ab}^{\gamma\delta} &= \mathbf{C} \epsilon_{ab} \epsilon^{\gamma\delta} & , \\ \mathbf{S}_{\alpha\beta}^{\gamma\delta} &= \mathbf{D} \delta_\alpha^\gamma \delta_\beta^\delta + \mathbf{E} \delta_\alpha^\delta \delta_\beta^\gamma & , & & \mathbf{S}_{\alpha\beta}^{cd} &= \mathbf{F} \epsilon_{\alpha\beta} \epsilon^{cd} & , \\ \mathbf{S}_{a\beta}^{c\delta} &= \mathbf{G} \delta_a^c \delta_\beta^\delta & , & & \mathbf{S}_{\alpha b}^{\gamma d} &= \mathbf{L} \delta_\alpha^\gamma \delta_b^d & , \\ \mathbf{S}_{a\beta}^{\gamma d} &= \mathbf{H} \delta_a^d \delta_\beta^\gamma & , & & \mathbf{S}_{\alpha b}^{c\delta} &= \mathbf{K} \delta_\alpha^\delta \delta_b^c & . \end{aligned} \quad (\text{D.2})$$

As described in [26], in the comparison between the worldsheet and the spin chain we are interested in the coefficients of $\mathcal{P}_g P_{pp'}^u S^B$, where \mathcal{P}_g is the graded permutation operator and $P_{pp'}^u$ exchanges the excitation momenta. Furthermore, to find the \mathbf{S} matrix for the full $PSU(2, 2|4)$ theory we use the relation

$$\mathbb{S} = \frac{1}{\mathbf{A}^B} \mathbf{S}^B \otimes \mathbf{S}^B \quad \mathbb{S}_{A\dot{A}B\dot{B}}^{C\dot{C}D\dot{D}}(p, p') = \frac{1}{\mathbf{A}^B} (\mathbf{S}^B)_{AB}^{CD}(p, p') (\mathbf{S}^B)_{\dot{A}\dot{B}}^{\dot{C}\dot{D}}(p, p') . \quad (\text{D.3})$$

This is because the $PSU(2|2)$ \mathbf{S} matrix was defined in [46] as the physical scattering matrix of the fields $\Phi_{A\dot{A}}$; in addition to \mathbf{S} for the left $PSU(2|2)$ indices, the scattering of this field receives contribution from $\mathbf{S}_{\dot{A}\dot{B}}^{\dot{C}\dot{D}} = A^B$.

Consequently we can relate the above coefficients to those of \mathbf{S} used in Beisert's \mathbf{S} matrix

$$\begin{aligned} \mathbf{A} &= \frac{1}{2\sqrt{\mathbf{A}^B}} (\mathbf{A}^B - \mathbf{B}^B) , & \mathbf{B} &= \frac{1}{2\sqrt{\mathbf{A}^B}} (\mathbf{A}^B + \mathbf{B}^B) , & \mathbf{C} &= \frac{i}{2\sqrt{\mathbf{A}^B}} \mathbf{C}^B , \\ \mathbf{D} &= \frac{1}{2\sqrt{\mathbf{A}^B}} (-\mathbf{D}^B + \mathbf{E}^B) , & \mathbf{E} &= \frac{1}{2\sqrt{\mathbf{A}^B}} (-\mathbf{D}^B - \mathbf{E}^B) , & \mathbf{F} &= \frac{i}{2\sqrt{\mathbf{A}^B}} \mathbf{F}^B , \\ \mathbf{G} &= \frac{1}{\sqrt{\mathbf{A}^B}} \mathbf{G}^B , & \mathbf{H} &= \frac{1}{\sqrt{\mathbf{A}^B}} \mathbf{H}^B , & & \\ \mathbf{L} &= \frac{1}{\sqrt{\mathbf{A}^B}} \mathbf{L}^B , & \mathbf{K} &= \frac{1}{\sqrt{\mathbf{A}^B}} \mathbf{K}^B . & & \end{aligned} \quad (\text{D.4})$$

The worldsheet perturbative expansion of the \mathbf{S} matrix is obtained from the spin-chain \mathbf{S} matrix in the small momentum expansion [40, 26], defined as the large- λ expansion at fixed string moment $p_{\text{string}} \sim \sqrt{\lambda} p_{\text{chain}}$:

$$p \longrightarrow \frac{2\pi p}{\sqrt{\lambda}} \quad p_{\text{chain}} = \frac{2\pi}{\sqrt{\lambda}} p_{\text{string}} = \frac{1}{\hat{g}} p_{\text{string}} . \quad (\text{D.5})$$

²⁸These definitions are similar but not identical to those of [46]. The relationship between the two definitions is given in equation D.4 below.

The matrix elements in (B.5) depend on $\hat{g} \propto 1/\sqrt{\lambda}$ only through x_p^\pm . In the small momentum expansion these variables (B.7) become ($\varepsilon = \sqrt{1+p^2}$):

$$x_p^\pm = \frac{1+\varepsilon}{p} \left(1 \pm \frac{i\pi p}{\sqrt{\lambda}} - \frac{\pi^2 p^2 (2+2\varepsilon+3p^2)}{6(1+\varepsilon+p^2)\sqrt{\lambda}^2} \mp \frac{i\pi^3 p^5}{6(1+\varepsilon+p^2)\sqrt{\lambda}^3} + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}^4}\right) \right) . \quad (\text{D.6})$$

$$x_p^\pm = \frac{1+\varepsilon}{p} \left(1 \pm \frac{ip}{2\hat{g}} - \frac{p^2(2+2\varepsilon+3p^2)}{24(1+\varepsilon+p^2)\hat{g}^2} \mp \frac{ip^5}{48(1+\varepsilon+p^2)\hat{g}^3} + \mathcal{O}\left(\frac{1}{\hat{g}^4}\right) \right) . \quad (\text{D.7})$$

An alternative parametrization of \mathbb{S} is obtained by extracting a phase factor from \mathbf{S} , that is $\mathbf{S} \mapsto \tilde{S}_0^{1/2} \tilde{\mathbf{S}}$. Then \mathbb{S} becomes

$$\mathbb{S} = \tilde{S}_0 \tilde{\mathbf{S}} \otimes \tilde{\mathbf{S}} ; \quad (\text{D.8})$$

The coefficients entering the tensor decomposition (D.2) are

$$\begin{aligned} \tilde{\mathbf{A}} &= \frac{x'_- - x_-}{x'_- - x_+} \frac{1 - \frac{1}{x'_- x_+}}{1 - \frac{1}{x'_+ x_+}} , \\ \tilde{\mathbf{B}} &= \frac{x'_+ - x_-}{x'_- - x_+} \left(1 - \frac{x'_- - x_-}{x'_+ - x_-} \frac{1 - \frac{1}{x'_- x_+}}{1 - \frac{1}{x'_+ x_+}} \right) , \\ \tilde{\mathbf{C}} &= \frac{i\gamma_p \gamma_{p'}}{x_+ x'_+} \frac{1}{1 - \frac{1}{x'_+ x_+}} \frac{x'_- - x_-}{x'_- - x_+} e^{\frac{ip'}{2}} , \\ \tilde{\mathbf{D}} &= \frac{x'_+ - x_+}{x'_- - x_+} \frac{1 - \frac{1}{x'_+ x_-}}{1 - \frac{1}{x'_- x_-}} e^{\frac{i(p'-p)}{2}} , \\ \tilde{\mathbf{E}} &= 1 - \frac{x'_+ - x_+}{x'_- - x_+} \frac{1 - \frac{1}{x'_+ x_-}}{1 - \frac{1}{x'_- x_-}} e^{\frac{i(p'-p)}{2}} , \\ \tilde{\mathbf{F}} &= -\frac{i(x_+ - x_-)(x'_+ - x'_-)}{\gamma_p \gamma_{p'} x_- x'_-} \frac{1}{1 - \frac{1}{x'_- x_-}} \frac{x'_+ - x_+}{x'_- - x_+} e^{-\frac{ip}{2}} , \\ \tilde{\mathbf{G}} &= \frac{x'_+ - x_+}{x'_- - x_+} e^{-\frac{ip}{2}} , \quad \tilde{\mathbf{H}} = \frac{\gamma_p}{\gamma_{p'}} \frac{x'_+ - x'_-}{x'_- - x_+} e^{\frac{i(p'-p)}{2}} , \\ \tilde{\mathbf{L}} &= \frac{x'_- - x_-}{x'_- - x_+} e^{\frac{ip'}{2}} , \quad \tilde{\mathbf{K}} = \frac{\gamma_{p'}}{\gamma_p} \frac{x_+ - x_-}{x'_- - x_+} , \end{aligned} \quad (\text{D.9})$$

with $x_\pm \equiv x_\pm(p)$, $x'_\pm \equiv x_\pm(p')$,

$$\gamma_p = |x_- - x_+|^{1/2}, \quad \gamma_{p'} = |x'_- - x'_+|^{1/2} .$$

and

$$\tilde{S}_0 = \frac{1 - \frac{1}{x'_+ x_-}}{1 - \frac{1}{x'_- x_+}} \frac{x'_- - x_+}{x'_+ - x_-} e^{i\theta(p,p')} . \quad (\text{D.10})$$

As before, p, p' stand for the spin-chain momenta.

E Two-particle cut at L -loops for string theory in $\text{AdS}_5 \times \text{S}^5$

We collect here the explicit forms s - and u -channel cuts of the L -loop four-point \mathbf{S} -matrix of the $\text{AdS}_5 \times \text{S}^5$ string. Here $L = L_1 + L_2 + 1$; the \mathbf{S} -matrix factor at L_1 -loops – *i.e.* the left factor in each term – carries the incoming particles).

- The s -channel two-particle cuts are:

$$iT^{(L)cd}_{ab}(p, p')|_{s\text{-cut}} = (A^{(L_1)}A^{(L_2)} + B^{(L_1)}B^{(L_2)} + 2C^{(L_1)}F^{(L_2)})\delta_a^c\delta_b^d + (A^{(L_1)}B^{(L_2)} + B^{(L_1)}A^{(L_2)} - 2C^{(L_1)}F^{(L_2)})\delta_a^d\delta_b^c \quad (\text{E.1})$$

$$iT^{(L)\gamma\delta}_{\alpha\beta}(p, p')|_{s\text{-cut}} = (D^{(L_1)}D^{(L_2)} + E^{(L_1)}E^{(L_2)} + 2F^{(L_1)}C^{(L_2)})\delta_\alpha^\gamma\delta_\beta^\delta + (D^{(L_1)}E^{(L_1)} + E^{(L_1)}D^{(L_2)} - 2F^{(L_1)}C^{(L_2)})\delta_\alpha^\delta\delta_\beta^\gamma \quad (\text{E.2})$$

$$iT^{(L)c\delta}_{a\beta}(p, p')|_{s\text{-cut}} = (H^{(L_1)}K^{(L_2)} + G^{(L_1)}G^{(L_2)})\delta_\beta^\delta\delta_a^c \quad (\text{E.3})$$

$$iT^{(L)\gamma d}_{a\beta}(p, p')|_{s\text{-cut}} = (G^{(L_1)}H^{(L_2)} + H^{(L_1)}L^{(L_2)})\delta_a^d\delta_\beta^\gamma \quad (\text{E.4})$$

$$iT^{(L)\gamma\delta}_{ab}(p, p')|_{s\text{-cut}} = ((A^{(L_1)} - B^{(L_1)})C^{(L_2)} + C^{(L_1)}(D^{(L_2)} - E^{(L_2)}))\epsilon_{ab}\epsilon^{\gamma\delta} \quad (\text{E.5})$$

$$iT^{(L)cd}_{\alpha\beta}(p, p')|_{s\text{-cut}} = (F^{(L_1)}(A^{(L_2)} - B^{(L_2)}) + (D^{(L_1)} - E^{(L_1)})F^{(L_2)})\epsilon_{\alpha\beta}\epsilon^{cd} \quad (\text{E.6})$$

$$iT^{(L)\gamma d}_{ab}(p, p')|_{s\text{-cut}} = (K^{(L_1)}H^{(L_2)} + L^{(L_1)}L^{(L_2)})\delta_a^d\delta_b^\gamma \quad (\text{E.7})$$

$$iT^{(L)c\delta}_{\alpha b}(p, p')|_{s\text{-cut}} = (K^{(L_1)}G^{(L_2)} + L^{(L_1)}K^{(L_2)})\delta_\alpha^\delta\delta_b^c \quad (\text{E.8})$$

A factor of $(i)^4$ was set to unity; the first $(i)^2$ is from the two cut propagators and a second factor of $(i)^2$ is due to the fact that scattering amplitudes are iT while the coefficients A, B, \dots parametrize T , cf. eq. (4.1).

- The u -channel two-particle cuts of an L -loop amplitude: as before, $L = L_1 + L_2 + 1$; the \mathbf{S} -matrix factor at L_1 -loop order – *i.e.* the left factor in each term – carries the first lower index and the second upper index:

$$iT^{(L)cd}_{ab}(p, p')|_{u\text{-cut}} = A^{(L_1)}A^{(L_2)}\delta_a^c\delta_b^d + (A^{(L_1)}B^{(L_2)} + B^{(L_1)}A^{(L_2)} + 2B^{(L_1)}B^{(L_2)} - 2H^{(L_1)}K^{(L_2)})\delta_a^d\delta_b^c \quad (\text{E.9})$$

$$iT^{(L)\gamma\delta}_{\alpha\beta}(p, p')|_{u\text{-cut}} = D^{(L_1)}D^{(L_2)}\delta_\alpha^\gamma\delta_\beta^\delta + (D^{(L_1)}E^{(L_2)} + E^{(L_1)}D^{(L_2)} + 2E^{(L_1)}E^{(L_2)} - 2K^{(L_1)}H^{(L_2)})\delta_\alpha^\delta\delta_\beta^\gamma \quad (\text{E.10})$$

$$iT^{(L)c\delta}_{a\beta}(p, p')|_{u\text{-cut}} = (-C^{(L_1)}F^{(L_2)} + G^{(L_1)}G^{(L_2)})\delta_\beta^\delta\delta_a^c \quad (\text{E.11})$$

$$iT^{(L)\gamma d}_{a\beta}(p, p')|_{u\text{-cut}} = (H^{(L_1)}D^{(L_2)} + A^{(L_1)}H^{(L_1)} + 2B^{(L_1)}H^{(L_2)} + 2H^{(L_1)}E^{(L_2)})\delta_a^d\delta_\beta^\gamma \quad (\text{E.12})$$

$$iT^{(L)\gamma\delta}_{ab}(p, p')|_{u\text{-cut}} = (G^{(L_1)}C^{(L_2)} + C^{(L_1)}L^{(L_2)})\epsilon_{ab}\epsilon^{\gamma\delta} \quad (\text{E.13})$$

$$iT^{(L)cd}_{\alpha\beta}(p, p')|_{u\text{-cut}} = (F^{(L_1)}G^{(L_2)} + L^{(L_1)}F^{(L_2)})\epsilon_{\alpha\beta}\epsilon^{cd} \quad (\text{E.14})$$

$$iT^{(L)\gamma d}_{\alpha b}(p, p')|_{u\text{-cut}} = (-F^{(L_1)}C^{(L_2)} + L^{(L_1)}L^{(L_2)})\delta_\alpha^\gamma\delta_b^d \quad (\text{E.15})$$

$$iT^{(L)c\delta}_{\alpha b}(p, p')|_{u\text{-cut}} = (K^{(L_1)}A^{(L_2)} + D^{(L_1)}K^{(L_2)} + 2K^{(L_1)}B^{(L_2)} + 2E^{(L_1)}K^{(L_2)})\delta_\alpha^\delta\delta_b^c \quad (\text{E.16})$$

As in the s -channel cuts, a factor of $(i)^4$ of the same origin as there was set to unity.

Clearly, to reconstruct the \mathbf{S} -matrix element one is to consider all choices of L_1 and L_2 such that $L_1 + L_2 = L - 1$. Generalized cuts can be constructed iteratively, by using (generalized) cuts in place of A^{L_i}, B^{L_i}, \dots

F One- and two-loop integrals

The one-loop massive bubble integral in two dimensions shown in fig. 2(a) and 2(b) were computed previously in [60] using the techniques of [66]; the third integral carried no momentum dependence. The first two integrals may be compactly written as

$$I(p, p') = \int \frac{d^2 q}{(2\pi)^2} \frac{1}{(q^2 + 1 + i\varepsilon)((q + p + p')^2 + 1 + i\varepsilon)} \quad (\text{F.1})$$

$$= \frac{i}{2\pi m^2} \frac{p_- p'_-}{p_-^2 - p'^-{}^2} \ln \left| \frac{p'_-}{p_-} \right| - \frac{p_- p'_-}{4m^2(p_- + p'_-)|p_- - p'_-|} \left(\frac{p_-}{|p_-|} + \frac{p'_-}{|p'_-|} \right) \quad (\text{F.2})$$

or, explicitly,

$$I(p, p') = \frac{i}{2\pi m^2} \frac{p_- p'_-}{p_-^2 - p'^-{}^2} \begin{cases} \ln \left(\frac{p'_-}{p_-} \right) - i\pi & \text{for } 0 < p_- < p'_- \text{ or } p'_- < p_- < 0 \\ \ln \left(-\frac{p'_-}{p_-} \right) & \text{for } p_- < 0 < p'_- \text{ or } p'_- < 0 < p_- \\ \ln \left(\frac{p'_-}{p_-} \right) + i\pi & \text{for } p_- < p'_- < 0 \text{ or } 0 < p'_- < p_- \end{cases} \quad (\text{F.3})$$

where

$$p_{\pm} = \frac{1}{2}(\varepsilon \pm p) \quad \text{and} \quad p_+ p_- = \frac{1}{4}. \quad (\text{F.4})$$

Assuming that $p_{\pm}, p'_{\pm} > 0$ and $p_- < p'_-$ the integrals that enter all amplitudes one- and two-loop amplitudes in worldsheet theories with excitations of *equal* masses are [44]:

$$\begin{aligned} I_s &= \frac{1}{J_s} \left(-\frac{i}{\pi} \ln \frac{p'_-}{p_-} - 1 \right) \\ I_u &= \frac{1}{J_u} \left(+\frac{i}{\pi} \ln \frac{p'_-}{p_-} + 0 \right) \\ I_t &= \frac{i}{4\pi} \\ I_a &= \left(\frac{1}{J_s} \left(-\frac{i}{\pi} \ln \frac{p'_-}{p_-} - 1 \right) \right)^2 \\ I_d &= \left(\frac{1}{J_u} \left(+\frac{i}{\pi} \ln \frac{p'_-}{p_-} + 0 \right) \right)^2 \\ I_b &= \frac{1}{16\pi^2} \left(\frac{4}{J_u^2} \ln^2 \frac{p'_-}{p_-} + \left(-\frac{8i\pi}{J_u^2} + \frac{2}{J_u} \right) \ln \frac{p'_-}{p_-} + \text{rational} \right) \\ I_c &= \frac{1}{16\pi^2} \left(\frac{4}{J_u^2} \ln^2 \frac{p'_-}{p_-} + \left(-\frac{8i\pi}{J_u^2} + \frac{2}{J_u} \right) \ln \frac{p'_-}{p_-} + \text{rational} \right) \\ I_e &= \frac{1}{16\pi^2} \left(\frac{4}{J_u^2} \ln^2 \frac{p'_-}{p_-} - \frac{2}{J_u} \ln \frac{p'_-}{p_-} + \text{rational} \right) \\ I_f &= \frac{1}{16\pi^2} \left(\frac{4}{J_u^2} \ln^2 \frac{p'_-}{p_-} - \frac{2}{J_u} \ln \frac{p'_-}{p_-} + \text{rational} \right) \end{aligned} \quad (\text{F.5})$$

Integrals with *different* masses assuming that $\frac{p}{m} > \frac{p'}{m'}$:

$$\tilde{I}_s = \frac{-i}{4\pi(p\varepsilon' - p'\varepsilon)} \left(\ln \left| \frac{p'_-}{p_-} \right| - \ln \left| \frac{m'}{m} \right| - i\pi \right) \quad (\text{F.6})$$

$$\tilde{I}_u = \frac{+i}{4\pi(p\varepsilon' - p'\varepsilon)} \left(\ln \left| \frac{p'_-}{p_-} \right| - \ln \left| \frac{m'}{m} \right| \right) \quad (\text{F.7})$$

where

$$\varepsilon = \sqrt{p^2 + m^2} . \quad (\text{F.8})$$

G s - and u -channel cuts of the one- and two-loop integrals

In the construction of the two-loop $C_{s,\text{extra}}$ and $C_{u,\text{extra}}$ integral coefficients it is necessary to take of the ansatz (3.6). We list here the two-particle cuts of the integrals that appear in the text.

$$\begin{aligned} I_s|_{s\text{-cut}} &= \frac{2}{4(p\varepsilon' - p'\varepsilon)} & I_s|_{u\text{-cut}} &= 0 \\ I_u|_{s\text{-cut}} &= 0 & I_u|_{u\text{-cut}} &= \frac{2}{4(p\varepsilon' - p'\varepsilon)} \\ I_a|_{s\text{-cut}} &= \frac{2 \times 2}{4(p\varepsilon' - p'\varepsilon)} I_s & I_a|_{u\text{-cut}} &= 0 \\ I_b|_{s\text{-cut}} &= \frac{1}{4(p\varepsilon' - p'\varepsilon)} (I_u + I_t) & I_b|_{u\text{-cut}} &= 0 \\ I_c|_{s\text{-cut}} &= \frac{1}{4(p\varepsilon' - p'\varepsilon)} (I_u + I_t) & I_c|_{u\text{-cut}} &= 0 \\ I_d|_{s\text{-cut}} &= 0 & I_d|_{u\text{-cut}} &= \frac{2 \times 2}{4(p\varepsilon' - p'\varepsilon)} I_u \\ I_e|_{s\text{-cut}} &= 0 & I_e|_{u\text{-cut}} &= \frac{1}{2(p\varepsilon' - p'\varepsilon)} (I_s + I_t) \\ I_f|_{s\text{-cut}} &= 0 & I_f|_{u\text{-cut}} &= \frac{1}{2(p\varepsilon' - p'\varepsilon)} (I_s + I_t) \end{aligned} \quad (\text{G.1})$$

In I_a and I_d one factor of 2 comes from the two chained bubbles and the second from the two solutions to the cut conditions.

H $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$ S-matrices

• The S matrix of Borsato, Ohlson Sax and Sfondrini

The S matrix proposed by Borsato, Ohlson Sax and Sfondrini assigns different amplitudes depending on whether the scattered states are a left-mover and a right-mover or are two of the same kind (where the left- and right-movers are representations of one of the other $PSU(1|1)$

factors of the $PSU(1|1)^2$ symmetry group of the gauge-fixed theory). There are also differences depending on the masses of the two states but for the most part this is contained within the Zhukowsky variables and we can write down the S matrix in terms of general coefficients without having to specify what the masses are. For the LL sectors it is defined as:

$$\mathbf{S}^{\text{BOSS}}|\phi\phi'\rangle = \mathbf{A}_{LL}^{\text{BOSS}}|\phi'\phi\rangle, \quad \mathbf{S}^{\text{BOSS}}|\phi\psi'\rangle = \mathbf{G}_{LL}^{\text{BOSS}}|\psi'\phi\rangle + \mathbf{H}_{LL}^{\text{BOSS}}|\phi'\psi\rangle, \quad (\text{H.1})$$

$$\mathbf{S}^{\text{BOSS}}|\psi\psi'\rangle = \mathbf{D}_{LL}^{\text{BOSS}}|\psi'\psi\rangle, \quad \mathbf{S}^{\text{BOSS}}|\psi\phi'\rangle = \mathbf{K}_{LL}^{\text{BOSS}}|\psi'\phi\rangle + \mathbf{L}_{LL}^{\text{BOSS}}|\phi'\psi\rangle, \quad (\text{H.2})$$

with the RR sectors behaving in a completely equivalent way. For the LR sectors the S matrix is defined as:

$$\mathbf{S}^{\text{BOSS}}|\phi\bar{\phi}'\rangle = \mathbf{A}_{LR}^{\text{BOSS}}|\bar{\phi}'\phi\rangle + \mathbf{C}_{LR}^{\text{BOSS}}|\bar{\psi}'\psi Z^-\rangle, \quad \mathbf{S}^{\text{BOSS}}|\phi\bar{\psi}'\rangle = \mathbf{G}_{LR}^{\text{BOSS}}|\bar{\psi}'\phi\rangle, \quad (\text{H.3})$$

$$\mathbf{S}^{\text{BOSS}}|\psi\bar{\psi}'\rangle = \mathbf{D}_{LR}^{\text{BOSS}}|\bar{\psi}'\psi\rangle + \mathbf{F}_{LR}^{\text{BOSS}}|\bar{\phi}'\phi Z^+\rangle, \quad \mathbf{S}^{\text{BOSS}}|\psi\bar{\phi}'\rangle = \mathbf{I}_{LR}^{\text{BOSS}}|\bar{\phi}'\psi\rangle, \quad (\text{H.4})$$

and again the RL sectors are similar to this.

For the L_1L_1 sector – *i.e.* the LL scattering of excitation of mass $m_1 = \alpha$ – the coefficients are given by:

$$\mathbf{A}_{L_1L_1}^{\text{BOSS}} = S_{L_1L_1} \frac{x_{p'}^+ - x_p^-}{x_{p'}^- - x_p^+}, \quad \mathbf{D}_{L_1L_1}^{\text{BOSS}} = -S_{L_1L_1}, \quad (\text{H.5})$$

$$\mathbf{G}_{L_1L_1}^{\text{BOSS}} = S_{L_1L_1} \frac{x_{p'}^+ - x_p^+}{x_{p'}^- - x_p^+}, \quad \mathbf{H}_{L_1L_1}^{\text{BOSS}} = S_{L_1L_1} \frac{x_{p'}^+ - x_{p'}^-}{x_{p'}^- - x_p^+} \frac{\eta_p}{\eta_{p'}}, \quad (\text{H.6})$$

$$\mathbf{K}_{L_1L_1}^{\text{BOSS}} = S_{L_1L_1} \frac{x_p^+ - x_p^-}{x_{p'}^- - x_p^+} \frac{\eta_{p'}}{\eta_p}, \quad \mathbf{L}_{L_1L_1}^{\text{BOSS}} = S_{L_1L_1} \frac{x_{p'}^- - x_p^-}{x_{p'}^- - x_p^+}. \quad (\text{H.7})$$

The R_1R_1 sector is exactly the same with $S_{L_1L_1} = S_{R_1R_1}$ and so are the L_2L_2/R_2R_2 sectors except there the mass appearing in Zhukowsky variables are different $m_2 = 1 - \alpha$ and the dressing phase factor could be different as well.

The coefficients of the L_1L_2 sectors S matrix – *i.e.* the LL scattering of excitation of different masses – are given by:

$$\mathbf{A}_{L_1L_2}^{\text{BOSS}} = S_{L_1L_2}, \quad \mathbf{D}_{L_1L_2}^{\text{BOSS}} = -S_{L_1L_2} \frac{x_{p'}^- - x_p^+}{x_{p'}^+ - x_p^-}, \quad (\text{H.8})$$

$$\mathbf{G}_{L_1L_2}^{\text{BOSS}} = S_{L_1L_2} \frac{x_{p'}^+ - x_p^+}{x_{p'}^+ - x_p^-}, \quad \mathbf{H}_{L_1L_2}^{\text{BOSS}} = S_{L_1L_2} \frac{x_{p'}^+ - x_{p'}^-}{x_{p'}^+ - x_p^-} \frac{\eta_p}{\eta_{p'}}, \quad (\text{H.9})$$

$$\mathbf{K}_{L_1L_2}^{\text{BOSS}} = S_{L_1L_2} \frac{x_p^+ - x_p^-}{x_{p'}^+ - x_p^-} \frac{\eta_{p'}}{\eta_p}, \quad \mathbf{L}_{L_1L_2}^{\text{BOSS}} = S_{L_1L_2} \frac{x_{p'}^- - x_p^-}{x_{p'}^+ - x_p^-}. \quad (\text{H.10})$$

Again the R_1R_2 sector is the same with $S_{L_1L_2} = S_{R_1R_2}$ and the L_2L_1/R_2R_1 sectors only differ by change of masses and dressing phase factor.

In the $L_1 R_1$ sector the coefficients are:

$$\mathbf{A}_{L_1 R_1}^{\text{BOSS}} = S_{L_1 R_1} \frac{1 - \frac{1}{x_p^+ x_{p'}^-}}{1 - \frac{1}{x_p^- x_{p'}^+}}, \quad \mathbf{C}_{L_1 R_1}^{\text{BOSS}} = -S_{L_1 R_1} \frac{\eta_p \eta_{p'}}{x_p^- x_{p'}^+} \frac{i}{1 - \frac{1}{x_p^- x_{p'}^+}}, \quad (\text{H.11})$$

$$\mathbf{D}_{L_1 R_1}^{\text{BOSS}} = -S_{L_1 R_1} \frac{1 - \frac{1}{x_p^- x_{p'}^+}}{1 - \frac{1}{x_p^+ x_{p'}^-}}, \quad \mathbf{F}_{L_1 R_1}^{\text{BOSS}} = S_{L_1 R_1} \frac{\eta_p \eta_{p'}}{x_p^+ x_{p'}^-} \frac{i}{1 - \frac{1}{x_p^+ x_{p'}^-}}, \quad (\text{H.12})$$

$$\mathbf{G}_{L_1 R_1}^{\text{BOSS}} = S_{L_1 R_1}, \quad \mathbf{L}_{L_1 R_1}^{\text{BOSS}} = S_{L_1 R_1} \frac{1 - \frac{1}{x_p^+ x_{p'}^-}}{1 - \frac{1}{x_p^- x_{p'}^+}}. \quad (\text{H.13})$$

The $L_2 R_1 / L_1 R_2 / L_2 R_2$ sectors behave in exactly the same way with the possibility of a different dressing phase.

The coefficients of the $R_1 L_1$ sector are given by:

$$\mathbf{A}_{R_1 L_1}^{\text{BOSS}} = S_{R_1 L_1} \frac{1 - \frac{1}{x_p^+ x_{p'}^-}}{1 - \frac{1}{x_p^- x_{p'}^+}}, \quad \mathbf{C}_{R_1 L_1}^{\text{BOSS}} = -S_{R_1 L_1} \frac{\eta_p \eta_{p'}}{x_p^- x_{p'}^+} \frac{i}{1 - \frac{1}{x_p^- x_{p'}^+}}, \quad (\text{H.14})$$

$$\mathbf{D}_{R_1 L_1}^{\text{BOSS}} = -S_{R_1 L_1} \frac{1 - \frac{1}{x_p^- x_{p'}^+}}{1 - \frac{1}{x_p^+ x_{p'}^-}}, \quad \mathbf{F}_{R_1 L_1}^{\text{BOSS}} = S_{R_1 L_1} \frac{\eta_p \eta_{p'}}{x_p^+ x_{p'}^-} \frac{i}{1 - \frac{1}{x_p^+ x_{p'}^-}}, \quad (\text{H.15})$$

$$\mathbf{G}_{R_1 L_1}^{\text{BOSS}} = S_{R_1 L_1} \frac{1 - \frac{1}{x_p^- x_{p'}^+}}{1 - \frac{1}{x_p^+ x_{p'}^-}}, \quad \mathbf{L}_{R_1 L_1}^{\text{BOSS}} = S_{R_1 L_1}. \quad (\text{H.16})$$

The $R_2 L_1 / R_1 L_2 / R_2 L_2$ sectors are the same with the possibility of different dressing phase.

There are further relations between the different scalar factors which are implied by the crossing equation [19]; we will not review them since they are not important for our calculation.

Phases can be added to make the S matrix satisfy the untwisted Yang-Baxter equations; it is this S matrix we will be using in the article:

$$\begin{aligned} \hat{\mathbf{A}}^{\text{BOSS}} &= \mathbf{A}^{\text{BOSS}} e^{\frac{i}{2}(p-p')} & \hat{\mathbf{C}}^{\text{BOSS}} &= \mathbf{C}^{\text{BOSS}} e^{-\frac{i}{2}p' + ib(p+p')} \\ \hat{\mathbf{D}}^{\text{BOSS}} &= \mathbf{D}^{\text{BOSS}} & \hat{\mathbf{F}}^{\text{BOSS}} &= \mathbf{F}^{\text{BOSS}} e^{\frac{i}{2}p - ib(p+p')} \\ \hat{\mathbf{G}}^{\text{BOSS}} &= \mathbf{G}^{\text{BOSS}} e^{-\frac{i}{2}p'} & \hat{\mathbf{H}}^{\text{BOSS}} &= \mathbf{H}^{\text{BOSS}} e^{\frac{i}{2}(p-p') + ib(p-p')} \\ \hat{\mathbf{K}}^{\text{BOSS}} &= \mathbf{K}^{\text{BOSS}} e^{-ib(p-p')} & \hat{\mathbf{L}}^{\text{BOSS}} &= \mathbf{L}^{\text{BOSS}} e^{\frac{i}{2}p} \end{aligned} \quad (\text{H.17})$$

The one-loop dressing phases in both mixed and unmixed sectors were extracted in [22] by comparing the one-loop corrections to the energy of classical string solutions with the predictions of the Asymptotic Bethe Ansatz:

$$\theta_{LL} = \theta(x_p, x_{p'}) = -\frac{\hat{\alpha}(x_p)\hat{\alpha}(x_{p'})}{\pi(x_p - x_{p'})^2} \ln \left(\frac{x_p + 1}{x_p - 1} \frac{x_{p'} - 1}{x_{p'} + 1} \right) + \text{rational}, \quad (\text{H.18})$$

$$\theta_{LR} = \tilde{\theta}(x_p, x_{p'}) = -\frac{\hat{\alpha}(x_p)\hat{\alpha}(x_{p'})}{\pi(1 - x_p x_{p'})^2} \ln \left(\frac{x_p + 1}{x_p - 1} \frac{x_{p'} - 1}{x_{p'} + 1} \right) + \text{rational}. \quad (\text{H.19})$$

The factors $\hat{\alpha}$ are:

$$\hat{\alpha}(x_p) = \frac{2m}{h} \frac{x^2}{x^2 - 1}, \quad x_p = \frac{p}{\varepsilon - m}. \quad (\text{H.20})$$

Here h plays the role of coupling constant. Its relation to the gauge theory coupling constant is not fixed by integrability or symmetries. At the level of the tree level worldsheet theory it is

$$h = \frac{\sqrt{\lambda}}{2\pi}; \quad (\text{H.21})$$

it potentially receives regularization scheme-dependent corrections and m is formally the mass parameter. Keeping m and m' and expanding in the small momentum limit, the one-loop phase factors are:

$$\theta_{LL} = \theta(x_p, x_{p'}) = \frac{-1}{2\pi h^2} \frac{p^2(p')^2[\mathbf{p} \cdot \mathbf{p}' + mm']}{(\varepsilon'p - p'\varepsilon)^2} \left(\ln \left(\frac{p'_-}{p_-} \right) - \ln \left(\frac{m'}{m} \right) \right) + \text{rational}, \quad (\text{H.22})$$

$$\theta_{LR} = \tilde{\theta}(x_p, x_{p'}) = \frac{-1}{2\pi h^2} \frac{p^2(p')^2[\mathbf{p} \cdot \mathbf{p}' - mm']}{(\varepsilon'p - p'\varepsilon)^2} \left(\ln \left(\frac{p'_-}{p_-} \right) - \ln \left(\frac{m'}{m} \right) \right) + \text{rational}. \quad (\text{H.23})$$

We notice that these phases correspond to twice the computed expressions in (6.16) and (6.19).

• The S matrix of Ahn and Bombardelli

The S matrix proposed in [18] is somewhat simpler in that it does not depend as much on what representations the scattered states are in.

The $L_1 L_1$ sector of the S matrix can be written as

$$\mathbf{S}^{\text{AB}}|\phi\phi'\rangle = \mathbf{A}_{LL}^{\text{AB}}|\phi'\phi\rangle, \quad \mathbf{S}^{\text{AB}}|\phi\psi'\rangle = \mathbf{G}_{LL}^{\text{AB}}|\psi'\phi\rangle + \mathbf{H}_{LL}^{\text{AB}}|\phi'\psi\rangle, \quad (\text{H.24})$$

$$\mathbf{S}^{\text{AB}}|\psi\psi'\rangle = \mathbf{D}_{LL}^{\text{AB}}|\psi'\psi\rangle, \quad \mathbf{S}^{\text{AB}}|\psi\phi'\rangle = \mathbf{K}_{LL}^{\text{AB}}|\psi'\phi\rangle + \mathbf{L}_{LL}^{\text{AB}}|\phi'\psi\rangle, \quad (\text{H.25})$$

with the coefficients given by:

$$\mathbf{A}_{L_1 L_2}^{\text{AB}} = S_{L_1 L_2}, \quad \mathbf{D}_{L_1 L_2}^{\text{AB}} = -S_{L_1 L_2} \frac{x_{p'}^- - x_p^+}{x_{p'}^+ - x_p^-}, \quad (\text{H.26})$$

$$\mathbf{G}_{L_1 L_2}^{\text{AB}} = S_{L_1 L_2} \frac{x_{p'}^+ - x_p^+}{x_{p'}^+ - x_p^-}, \quad \mathbf{H}_{L_1 L_2}^{\text{AB}} = S_{L_1 L_2} \frac{x_{p'}^+ - x_{p'}^-}{x_{p'}^+ - x_p^-} \frac{\omega_p}{\omega_{p'}}, \quad (\text{H.27})$$

$$\mathbf{K}_{L_1 L_2}^{\text{AB}} = S_{L_1 L_2} \frac{x_p^+ - x_p^-}{x_{p'}^+ - x_p^-} \frac{\omega_{p'}}{\omega_p}, \quad \mathbf{L}_{L_1 L_2}^{\text{AB}} = S_{L_1 L_2} \frac{x_{p'}^- - x_p^-}{x_{p'}^+ - x_p^-}. \quad (\text{H.28})$$

where ω_p and $\omega_{p'}$ are chosen to be 1 and S_{XY} are dressing phases. The $L_2L_2/R_1R_1/R_2R_2$ sectors are exactly the same with the appropriate changes of masses. The $L_1R_1/L_2R_1/R_1L_2/R_2L_1$ are also similar, but with a different dressing phase, while the $L_1L_2/L_2L_1/R_1R_2/R_2R_1$ and the $L_1R_2/L_2R_1/R_1L_2/R_2L_1$ sectors are also the same except that the dressing phase is set to be 1.

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